Croissance du volume des boules dans les revêtements universels des graphes et surfaces.

Steve Karam

UNIVERSITÉ FRANÇOIS-RABELAIS DE TOURS

École Doctorale Mathématiques, Informatiques, Physique théoriques et Ingénierie de systèmes

Laboratoire de Mathématiques et Physique Théorique

THÈSE présenté par :

Steve KARAM

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CROISSANCE DU VOLUME DES BOULES DANS LES REVÊTEMENTS UNIVERSELS DES GRAPHES ET DES SURFACES

THÈSE DIRIGÉE PAR :

M. SABOURAU Stéphane	Professeur, Université Paris-Est, Créteil

RAPPORTEURS :

M. BALACHEFF Florent	Maître de conférences HDR, Université Lille 1
M. BAVARD Christophe	Professeur, Université Bordeaux 1

JURY :

M. BALACHEFF Florent	Maître de conférences HDR, Université Lille 1
M. BAVARD Christophe	Professeur, Université Bordeaux 1
M. COURTOIS Gilles	Directeur de Recherche, CNRS
M. EL SOUFI Ahmad	Professeur, Université François-Rabelais, Tours
M. PEIGNÉ Marc	Professeur, Université François-Rabelais, Tours
M. SABOURAU Stéphane	Professeur, Université Paris-Est, Créteil

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Résumé

Dans le cadre de la géométrie riemannienne globale sans hypothèse de courbure en lien avec la topologie, nous nous intéressons au volume maximal des boules de rayon fixé dans les revêtements universels des graphes et des surfaces.

Dans la première partie, nous prouvons que si l'aire d'une surface riemannienne fermée M de genre $g \geq 2$ est suffisamment petite par rapport à son aire hyperbolique, alors pour chaque rayon $R \geq 0$, le revêtement universel de M contient une R-boule d'aire au moins l'aire d'une cR-boule dans le plan hyperbolique, où $c \in (0, 1)$ est une constante universelle. En particulier (quitte à prendre l'aire de la surface encore plus petite), nous démontrons que pour chaque rayon $R \geq 1$, le revêtement universel de M contient une R-boule d'aire au moins l'aire d'une R-boule dans le plan hyperbolique. Ce résultat répond positivement pour les surfaces, à une question de L. Guth. Nous démontrons également que si Γ est un graphe connexe de premier nombre de Betti $b \geq 2$ et de longueur suffisamment petite par rapport à la longueur d'un graphe trivalent Γ_b de premier nombre de Betti b dont la longueur de chaque arête est 1, alors pour chaque rayon $R \geq 0$, le revêtement universel de Γ contient une R-boule d'aire au moins c fois l'aire d'une R-boule dans le revêtement universel de Γ_b, où $c \in (\frac{1}{2}, 1)$.

Dans la deuxième partie, nous généralisons un théorème de M. Gromov concernant le nombre maximal de courts lacets homotopiquement indépendants basés en un même point. Plus précisément, nous prouvons que sur toute surface riemannienne fermée M de genre $g \geq 2$ et d'aire normalisée à g, il existe au moins $\lceil \log(2g) + 1 \rceil$ lacets homotopiquement indépendants basés en un même point de longueur au plus $C \log(g)$, où C est une constante positive indépendante du genre. Comme corollaire immédiat de ce théorème, nous redémontrons l'inégalité systolique asymptotique sur la systole séparante. Nous démontrons également un théorème analogue pour les graphes métriques. Plus précisément, nous prouvons que sur chaque graphe métrique Γ de premier nombre de Betti $b \geq 2$ et de longueur b, il existe au moins $\lfloor \log(b) \rfloor$ lacets homologiquement indépendants basés en un même point de longueur au plus $48 \log(b)$. Ce résultat étend la borne en $\log(b)$ sur la systole homologique dûe à Bollobàs-Szemerédi-Thomason à au moins $\log(b)$ lacets homologiquement indépendants basés en un même point. En outre, nous donnons des exemples de graphes où notre résultat est optimal (à une constante multiplicative près).

Mots clés : Surface, graphe, revêtement universel, entropie, aire des boules, systole, lacets homotopiquement indépendants, inégalités géometriques.

Abstract

This thesis deals with global Riemannian geometry without curvature assumptions and its link to topology, we focus on the maximal volume of balls of fixed radius in the universal covers of graphs and surfaces.

In the first part, we prove that if the area of a closed Riemannian surface M of genus at least two is sufficiently small with respect to its hyperbolic area, then for every radius $R \geq 0$ the universal cover of M contains an R-ball with area at least the area of a cR-ball in the hyperbolic plane, where $c \in (0, 1)$ is a universal positive constant. In particular (taking the area of M smaller if needed), we prove that for every radius $R \geq 1$, the universal cover of M contains an R-ball with area at least the area of a ball with the same radius in the hyperbolic plane. This result answers positively a question of L. Guth for surfaces. We also prove an analog result for graphs. Specifically, we prove that if Γ is a connected metric graph of first Betti number $b \geq 2$ and of length sufficiently small with respect to the length of a connected trivalent graph Γ_b of the same Betti number where the length of each edge is 1, then for every radius $R \geq 0$ the universal cover of Γ contains an R-ball with length at least c times the length of an R-ball in the universal cover of Γ_b, where $c \in (\frac{1}{2}, 1)$ is a universal constant.

In the second part, we generalize a theorem of M. Gromov concerning the maximal number of homotopically independentshort loops based at the same point . Specifically, we prove that on every closed Riemannian surface M of genus $g \geq 2$ and area normalized to g there exist at least $\lceil \log(2g) + 1 \rceil$ homotopically independent loops based at the same point of length at most $C \log(g)$, where C is some positive constant independent from the genus. As an immediate corollary of this theorem, we recapture the asymptotic systolic inequality on the separating systole. We also prove a similar theorem for metric graphs. Precisely, we prove that on every metric graph Γ of first Betti number $b \geq 2$ and length b, there exist at least $\lfloor \log(b) \rfloor$ homologically independent loops based at the same point of length at most $48 \log(b)$. That extends Bollobàs-Szemerédi-Thomason's $\log(b)$ bound on the homological systole to at least $\log(b)$ homologically independent loops based at the same point. Moreover, we give examples of graphs where our result is optimal (up to a multiplicative constant).

Keywords : Surface, graph, universal cover, entropy, area of balls, systole, homologically independent loops, geometric inequalities.

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Introduction

Introduction et présentation des résultats

0.1 Présentation générale

La géométrie riemannienne sans hypothèse de courbure et les liens avec la topologie sont à la base de ce travail. Nous nous intéressons au volume maximal des boules de rayon fixé dans les revêtements universels des graphes et des surfaces. Cet invariant est lié à la géométrie asymptotique des revêtements universels et à celle des groupes fondamentaux ainsi qu'à l'entropie du flot géodésique.

Le but principal de cette thèse est de démontrer le résultat suivant. Pour tout rayon $R \geq 1$, il existe dans le revêtement universel d'une surface fermée de genre au moins deux et d'aire relativement petite par rapport au genre, une boule de rayon R d'aire au moins l'aire d'une boule de même rayon dans l'espace hyperbolique. Nous démontrons également un théorème analogue pour les graphes. Dans une autre direction, nous obtenons une estimée sur le nombre maximal de courts lacets basés en un même point homotopiquement indépendants dans une surface de genre au moins deux et d'aire égale à l'aire hyperbolique.

Soit $(\widetilde{M}, \widetilde{g})$ le revêtement universel d'une variété riemannienne fermée (M, g). On définit la fonction

$$V_{(\widetilde{M},\widetilde{g})}(R) := \sup_{\tilde{x} \in \widetilde{M}} \operatorname{Vol} B_{\widetilde{g}}(\tilde{x}, R),$$

représentant le plus grand volume d'une boule de rayon R dans $(\widetilde{M}, \widetilde{g})$. Lorsque R tend vers l'infini, cet invariant décrit la géométrie asymptotique du revêtement universel de M. Il est aussi lié à la géométrie du groupe fondamental de M et à l'entropie du flot géodésique sur M.

Lorsque la courbure de (M, g) est majorée par une constante strictement négative, l'inégalité de Bishop-Gromov-Gunther fournit un minorant de type exponentiel de la fonction $V(R)$. Dans ce travail, nous cherchons à minorer la fonction $V(R)$ sans imposer de contrôle sur la courbure ce qui est une hypothèse locale forte. Nous remplaçons ce contrôle par une hypothèse topologique et un contrôle sur le volume de (M, g). Plus précisément, nous supposons que M est une variété de type hyperbolique ($i.e.$, sur laquelle il existe une métrique hyperbolique) et nous remplaçons la majoration de l'invariant local de courbure par une majoration de l'invariant global de volume.

À notre connaissance, seuls trois résultats existent dans cette direction. Le premier résultat est dû à M. Gromov.

Théorème 0.1.1 ([14], page 37). *Soit un entier $n \geq 2$. Il existe une constante c_n telle que si (M, h) une variété hyperbolique fermée de dimension n et g est une autre métrique sur M avec $\operatorname{Vol}(M, g) < \operatorname{Vol}(M, h)$ alors il existe un rang R_0 (dépendant de g) tel que pour tout rayon $R > R_0$ on a*

$$V(R) \geq V_{\mathbb{H}^n}(c_n R),$$

où $\mathbb{H}^n$ est l'espace hyperbolique de dimension n.

En fait, ce résultat est encore valable pour toute variété riemannienne fermée de volume simplicial non nul, si on remplace le volume hyperbolique $\mathrm{Vol}(M,h)$ par le volume simplicial.
G. Besson, G. Courtois, S. Gallot ont demontré le résultat de Gromov avec une constante c_n optimale.

Théorème 0.1.2 ([2]). *Soit un entier $n \geq 2$. Si (M,h) est une variété hyperbolique fermée de dimension n et g est une autre métrique sur M avec $\mathrm{Vol}(M,g) < \mathrm{Vol}(M,h)$, alors il existe un rang R_0 (dépendant de g) tel que pour tout rayon $R \geq R_0$, on a*

$$V(R) \geq V_{\mathbb{H}^n}(R).$$

Dans [16], L. Guth a cherché les valeurs uniformes de R, c'est-à-dire ne dépendant pas de la métrique, pour lesquels l'inégalité $V(R) \geq V_{\mathbb{H}^n}(R)$ du théorème 0.1.2 reste valable. Il a démontré que si le volume de (M,g) est suffisamment petit par rapport au volume hyperbolique alors l'inégalité est vraie pour $R = 1$. Spécifiquement, il a démontré le résultat suivant.

Théorème 0.1.3 ([16]). *Soit un entier $n \geq 2$. Il existe une constante $\delta_n \in (0,1)$ telle que que si (M,h) est une variété hyperbolique fermée de dimension n et g est une autre métrique sur M avec $\mathrm{Vol}(M,g) \leq \delta_n \mathrm{Vol}(M,h)$ alors*

$$V_{(\widetilde{M},\widetilde{g})}(1) \geq V_{\mathbb{H}^n}(1).$$

Remarquons que dans le théorème 0.1.3, nous avons $\mathrm{Vol}(M,g) < \mathrm{Vol}(M,h)$. Donc par le théorème 0.1.2, il existe un rang R_0 qui dépend de la métrique à partir duquel l'inégalite $V_{(\widetilde{M},\widetilde{g})}(R) \geq V_{\mathbb{H}^n}(R)$ est satisfaite. Cette observation a poussé L. Guth à se demander si cette inégalité est encore verifiée pour R compris entre 1 et R_0. Plus précisément, il pose la question suivante.

(Q1) : *Existe-t-il une constante $\delta_n > 0$ telle que si (M,h) est une variété hyperbolique fermée de dimension n et g est une autre métrique sur M avec $\mathrm{Vol}(M,g) \leq \delta_n \mathrm{Vol}(M,h)$ alors pour tout $R \geq 1$, on a*

$$V_{(\widetilde{M},\widetilde{g})}(R) \geq V_{\mathbb{H}^n}(R).$$

Nous nous sommes intéressés à la question de L. Guth dans cette thèse et avons obtenu plusieurs résultats dans cette direction.

Dans le premier chapitre de la thèse, nous répondons positivement à la question (Q1) dans le cas des surfaces. Nous démontrons aussi un théorème analogue pour les graphes.

Commençons par énoncer nos résultats dans le cadre des surfaces.

Théorème A. *Il existe deux constantes positives δ et c telles que si (M, h) est une surface hyperbolique fermée et g est une autre métrique sur M avec $\mathrm{aire}(M, g) \leq \delta\, \mathrm{aire}(M, h)$ alors il existe un point x dans $\widetilde{M}$ tel que pour tout $R \geq 0$, on a*

$$\mathrm{aire}\, B_{(\widetilde{M},\widetilde{g})}(x, R) \geq V_{\mathbb{H}^2}(cR).$$

En particulier

$$V_{(\widetilde{M},\widetilde{g})}(R) \geq V_{\mathbb{H}^2}(cR).$$

En prenant l'aire de (M, g) suffisamment petite par rapport à l'aire hyperbolique dans le théorème A, nous pouvons répondre positivement à la question (Q1) puisque dans ce cas, et pour les valeurs $R \geq 1$, nous pouvons prendre $c = 1$. Plus précisément, nous avons

Theorem A'. *Il existe une constante positive δ telle que si (M, h) est une surface hyperbolique fermée et g est une autre métrique sur M avec $\mathrm{aire}(M, g) \leq \delta\, \mathrm{aire}(M, h)$ alors il existe un point x dans $\widetilde{M}$ tel que pour tout $R \geq 1$, on a*

$$\mathrm{aire}\, B_{(\widetilde{M},\widetilde{g})}(x, R) \geq V_{\mathbb{H}^2}(R).$$

En particulier

$$V_{(\widetilde{M},\widetilde{g})}(R) \geq V_{\mathbb{H}^2}(R).$$

Notons que le théorème A découle d'un résultat analogue pour les graphes. Avant de l'énoncer, introduisons quelques définitions.

La fonction $V(R)$ que nous avons définie dans le cadre des variétés riemanniennes possède un analogue pour les graphes métriques. Rappelons qu'un graphe métrique (Γ, h) est un CW-complexe de dimension 1 muni d'une distance h telle que Γ est un espace de longueur (Pour plus de details sur les graphes, nous invitons le lecteur à consulter [8]). Notons $(\widetilde{\Gamma}, \widetilde{h})$ le revêtement universel de (Γ, h). Nous définissons la fonction

$$V'(R) = \sup_{\widetilde{v} \in \widetilde{\Gamma}} \mathrm{longueur}(B_{\widetilde{h}}(\widetilde{v}, R)),$$

où par “longueur” on désigne la mesure de Hausdorff 1-dimensionnelle associée à la métrique $\widetilde{h}$.

Un graphe k-régulier est un graphe où tous les sommets ont la même valence ou degré k. Pour tout entier $b \geq 2$, on note par Γ_b un graphe connexe trivalent (3-régulier) de premier nombre de Betti b et par h_b la métrique sur Γ telle que la longueur de chaque arête est égale à 1.

Pour notre problème, les graphes Γ_b sont les analogues des variétés hyperboliques. En effet, l'analogue du théorème 0.1.2 pour les graphes, établi par I. Kapovich, et T. Nagnibeda, s'énonce à l'aide des graphes Γ_b comme suit.

Théorème 0.1.4 ([19]). *Soit* (Γ, h) *un graphe métrique connexe de premier nombre de Betti* $b \geq 2$, *avec* longueur$(\Gamma, h) <$ longueur(Γ_b, h_b). *Alors il existe une valeur* R_0 *tel que pour tout* $R \geq R_0$ *on a*

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

À la lumière des théorèmes 0.1.3 et 0.1.4, il est naturel de poser la question suivante, analogue de la question (Q1) pour les graphes.

(Q2) : *Existe-t-il une constante* $\delta > 0$ *telle que si* longueur$(\Gamma, h) < \delta$ longueur(Γ_b, h_b) *alors pour tout* $R \geq 0$, *on a*

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

Nous faisons un premier pas vers une réponse à la question (Q2) avec le résultat suivant.

Théorème B. *Soit* $\delta \in \left]0, \frac{1}{6}\right]$. *Si* (Γ, h) *est un graphe métrique connexe de premier nombre de Betti* $b \geq 2$ *tel que* longueur$(\Gamma, h) \leq \delta$ longueur(Γ_b, h_b) *alors il existe un point* x *dans* $\widetilde{\Gamma}$ *tel que pour tout* $R \geq 0$, *on a*

$$\text{longueur}\, B_{(\widetilde{\Gamma},\widetilde{h})}(x, R) \geq (1 - 3\delta) V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

En particulier

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq (1 - 3\delta) V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

En comparant les théorèmes A et B, nous remarquons que sous des conditions analogues, la constante $c' = 1 - 3\delta$ dans la conclusion du théorème B est multiplicative alors que la constante c dans celle du théorème A intervient dans l'exponentielle du volume des boules de $\mathbb{H}^2$. Nous en déduisons que l'inégalité du théorème B est de nature plus forte que celle du théorème A.

Nous indiquons à présent dans les grandes lignes comment le théorème A peut se déduire du théorème B. Fixons $R \geq 0$. Tout d'abord, nous montrons que nous pouvons supposer que la systole homotopique de (M, g), notée $\text{sys}(M, g)$ et définie comme la longueur du plus court lacet non-contractile de M, est au moins $\max\{2R, 1/2\}$. Nous considérons ensuite un graphe connexe Γ plongé dans M, capturant la topologie de M (c'est-à-dire tel que l'inclusion de Γ dans M induit un isomorphisme entre $H_1(\Gamma, \mathbb{Z})$ et $H_1(M, \mathbb{Z})$), de longueur minimale. En utilisant la minoration de la systole et la borne sur l'aire de (M, g), nous montrons que la longueur de Γ satisfait l'hypothèse du théorème B, disons pour $\delta = \frac{1}{6}$. Par conséquent, il existe un point x dans $\widetilde{\Gamma}$ tel que pour tout rayon $r \in (0, R)$, la longueur de la boule $B_{\widetilde{\Gamma}}(r)$ centrée en x et de rayon r dans $\widetilde{\Gamma}$ croît de manière exponentielle. Plus précisément, elle est supérieure ou égale à $\frac{1}{2} V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R)$. Puisque $R \leq \frac{1}{2}\text{sys}(\Gamma, h)$, la longueur de la projection $B_\Gamma(r)$ de $B_{\widetilde{\Gamma}}(r)$ dans Γ coïncide avec la longueur de $B_{\widetilde{\Gamma}}(r)$. Considérons maintenant la boule $B_M(r)$ de rayon r dans M concentrique à la boule $B_\Gamma(r)$. Pour tout $r \leq R$, la longueur de $\partial B_M(r)$ est supérieure ou égale à la longueur de l'intersection $\Gamma \cap B_M(r)$, autrement nous pourrions construire un autre graphe Γ' capturant la topologie de M mais de longueur plus courte que Γ. Ce serait en contradiction avec la minimalité de Γ. Comme $\Gamma \cap B_M(r)$ contient $B_\Gamma(r)$, nous en déduisons que la longueur de $\partial B_M(r)$ est supérieure ou égale à celle de $B_\Gamma(r)$. Par la formule de la co-aire, nous concluons que l'aire de la boule $B_M(R)$ croît de manière exponentielle. Pour terminer la preuve, notons que puisque le diamètre de $B_M(R)$ est inférieur à la systole de M, on a $V_{(\widetilde{M},\widetilde{g})}(R) = V_{(M,g)}(R)$.

Dans le reste de cette introduction, nous allons adopter une approche indirecte pour minorer la fonction $V(R)$. Spécifiquement, nous nous intéressons à la question suivante.

(Q3) : *Étant donnée une surface M de genre au moins deux et d'aire égale à l'aire hyperbolique, quel est le nombre maximal k de lacets homotopiquement indépendants basés en un même point de longueur au plus $\sim \log(\text{aire}(M,g))$?*

Ici, k lacets de M basés en même point sont dits homotopiquement indépendants si leurs classes d'homotopie engendrent un sous-groupe libre de rang k dans le groupe fondamental de M.
Avant d'aller plus loin, mentionnons quelques motivations derrière la question (Q3).

1. Une réponse à la question (Q3) permet (sous certaines conditions) de minorer la fonction $V(R)$ pour les grands rayons R. Cette idée est expliquée en détail à la fin de l'introduction.
2. Une réponse même partielle à la question (Q3) (spécifiquement montrant que $k \geq 2$), permet de redémontrer un théorème de S. Sabourau sur la systole séparante, *i.e.*, la longueur du plus court lacet non-contractile trivial en homologie et de raffiner sa preuve. Cette idée est expliquée plus loin dans l'introduction.
3. La question (Q3) et ses ramifications nous paraissent également intéressantes en soi. Gromov s'est beaucoup intéressé à cette question dans [10] où il a obtenu les premiers résultats sur le sujet. Mentionnons aussi que Balacheff-Parlier-Sabourau ont répondu à cette question dans [1] pour des lacets homologiquement indépendants mais pas forcément basés en un même point .

Dans le deuxième chapitre de cette thèse nous traitons de la question (Q3) et d'une question analogue pour les graphes. Commençons par énoncer un premier résultat dans le cadre des surfaces.

Dans ce qui suit pour un nombre réel positif R, nous notons par $\lceil R \rceil$ le plus petit entier supérieur ou égal à R.

Théorème C. *Soit M une surface riemannienne fermée de genre $g \geq 2$ et d'aire normalisée à g. Il existe au moins $\lceil \log(2g)+1 \rceil$ lacets homotopiquement indépendants $\gamma_1, \dots \gamma_{\lceil \log(2g)+1 \rceil}$ basés en un même point dans M tels que*

$$\text{longueur}(\gamma_i) \leq C \log(g),$$

où C est une constante positive universelle indépendante du genre et de la métrique.

Le théorème C améliore considérablement la seule réponse existante (jusqu'à présent) à la question (Q3). Cette réponse établie par M. Gromo prend la forme suivante.

Théorème 0.1.5. *([10]) Soit M une surface riemannienne fermée de genre $g \geq 2$ et d'aire normalisée à g. Pour chaque $\alpha < 1$, il existe deux lacets γ_1 et γ_2 homotopiquement indépendants basés en un même point de M, tels que*

$$\text{longueur}(\gamma_i) \leq C_\alpha \, g^{1-\alpha}.$$

où C_α est une constante positive qui ne dépend que de α.

Sous les mêmes hypothèses que le théorème 0.1.5, le théorème C garantit l'existence de $\lceil \log(2g) + 1 \rceil$ (au lieu de 2) lacets homotopiquement indépendants basés en un même point de longeur au plus $\sim \log(g)$ (au lieu de $g^{1-\alpha}$).

D'autre part, le théorème C permet de redémontrer le théorème suivant de S. Sabourau au moyen d'une preuve alternative.

Théorème 0.1.6 ([25])**.** *Il existe une constante positive C telle que toute surface riemannienne fermée M de genre $g \geq 2$ et d'aire g satisfait*

$$\mathrm{sys}_0(M) \leq C \log(g),$$

où $\mathrm{sys}_0(M)$ est la systole séparante de M.

Soulignons que S. Sabourau commence sa preuve en supposant que $\mathrm{sys}_0(M) \geq 4\,\mathrm{sys}(M)$ puisque dans le cas contraire, le résultat découle de l'inégalite systolique de Gromov. Le théorème C fournit en fait une preuve uniforme de l'inégalite systolique asymptotique sur la systole séparante n'utilisant pas l'inégalité systolique de Gromov. Pour déduire le théorème 0.1.6 du théorème C, il suffit de considérer le commutateur de n'importe quelle paire de lacets du théorème C. Ce commutateur est homologiquement mais non homotopiquement trivial et sa longueur est au plus $4C\log(g)$. On en déduit immédiatement une borne sur la systole séparante.

Dans la deuxième partie du deuxième chapitre, nous répondons à une question analogue à la question (Q3) pour les graphes. Spécifiquement nous répondons à la question suivante.

(Q4) : *Étant donné un graphe métrique (Γ, h) de premier nombre de Betti $b \geq 2$ et de longueur b, quel est le nombre maximal de lacets homotopiquement indépendants basés en un même point de longueur au plus $\sim \log(b)$?*

Nous démontrons le résultat suivant.

Théorème D. *Soit Γ un graphe métrique connexe de premier nombre de Betti $b \geq 2$ et de longueur b. Étant donné $n \in \{1, \ldots, b\}$, il existe au moins n lacets homologiquement indépendants $\gamma_1 \ldots, \gamma_n$ dans Γ basés en un même point tels que*

$$\text{longueur}(\gamma_i) \leq 24(\log(b) + n).$$

En outre, nous montrons que le théorème D est optimal (à une constante multiplicative près). Ainsi nous obtenons une réponse complète à la question (Q4). Notons que la seule réponse connue à la question (Q4) avant le théorème D est le résultat suivant de Bollobàs-Szemerédi-Thomason sur la systole des graphes métriques.

Théorème 0.1.7 ([3], [4]). *Soit* (Γ, h) *un graphe métrique de premier nombre de Betti* $b \geq 2$ *et de longueur* b. *Il existe un lacet* γ *homotopiquement non-trivial tel que*

$$\text{longueur}(\gamma) \leq 4 \log(b+1).$$

Le théorème D étend considérablement le théorème 0.1.7. Sous les même hypothèses que le théorème 0.1.7, le théorème D garantit l'existence de $\lfloor \log(b) \rfloor$ (au lieu d'un seul) lacets homologiquement indépendants dans Γ basés en un même point et de longueur au plus $\sim \log(b)$.

Nous terminons cette introduction en expliquant comment une réponse à la question (Q3) en toute dimension peut éventuellement fournir un minorant de la fonction $V(R)$ pour les grands rayons R.

Soit (M, g) une variété riemannienne fermée de type hyperbolique. Supposons qu'il existe un système S de k lacets homotopiquement indépendants basés en un même point m de M. En outre, supposons que le volume de la boule de M centrée en m et de rayon s est minorée par disons 1, où s est la moitié de la systole basée en m. Pour R assez grand, considérons la boule $B = B_{\widetilde{g}}(\widetilde{m}, R-s)$ dans le revêtement universel $\widetilde{M}$ de M centrée en $\widetilde{m}$ de rayon $R - s$. Le nombre $L = \max_{\beta \in S} \text{longueur}(\beta)$ permet d'estimer la croissance exponentielle de l'orbite de $\widetilde{m}$ par l'action du sous groupe libre $< S >$ de rang k. Il fournit un minorant du nombre de points de cette orbite contenues dans B. Les boules de rayon s centrées en les points de l'orbite de $\widetilde{m}$ étant disjointes et de volume minoré, nous en déduisons que le volume de la boule $B_{\widetilde{g}}(\widetilde{m}, R)$ croît de manière exponentielle au moins comme $\sim e^{\frac{R}{L} \log(k)}$. Pour plus de détails sur cette idée nous invitons le lecteur à consulter la section 2 du chapitre 2.

Bien que partageant un thème commun, les deux parties de la thèse sont indépendantes et peuvent être lues comme telles.

Chapitre 1

Volumes des boules dans les revêtements universels des graphes et surfaces

Growth of balls in the universal cover of surfaces and graphs

Abstract

In this paper, we prove uniform lower bounds on the volume growth of balls in the universal covers of Riemannian surfaces and graphs. More precisely, there exists a constant $\delta > 0$ such that if (M, hyp) is a closed hyperbolic surface and h another metric on M with $\mathrm{Area}(M,h) \leq \delta\,\mathrm{Area}(M, hyp)$ then for every radius $R \geq 1$ the universal cover of (M,h) contains an R-ball with area at least the area of an R-ball in the hyperbolic plane. This positively answers a question of L. Guth for surfaces. We also prove an analog theorem for graphs.

1.1 Introduction

Let $(\widetilde{M}, \widetilde{h})$ be the universal cover of a closed Riemannian manifold (M,h). We consider the function

$$V_{(\widetilde{M},\widetilde{h})}(R) := \sup_{\tilde{x}\in\widetilde{M}} \mathrm{Vol}\, B_{\widetilde{h}}(\tilde{x}, R).$$

The function $V(R)$ is the largest volume of any ball of radius R in $(\widetilde{M}, \widetilde{h})$. Since it is possible to construct examples of Riemannian manifolds where the volume of some balls of radius R in the universal cover is arbitrary small, it is interesting to know whether there is at least one ball of radius R in the universal cover with a large volume. If the curvature of the metric h is bounded above by a negative constant then the Bishop-Gunther-Gromov inequality gives us an exponential lower bound on the volume of all balls in the universal cover $\widetilde{M}$. So in particular we have an estimate of the function V. In this paper, we are interested in finding curvature-free exponential lower bounds for V. We replace the local assumption, namely a curvature bound, by a topological assumption and a condition on the volume of (M,h). What is believed is that if the topology of M is complicated then the function V is large (see [11] and [16] for more details).

Before going further, we would like to point out that the function $V_{(\widetilde{M},\widetilde{h})}$ is related to the volume entropy of (M,h). The volume entropy of (M,h) is defined as

$$\mathrm{Ent}(M,h) = \lim_{R\to+\infty} \frac{\log(Vol(B_{\widetilde{h}}(\tilde{x}, R)))}{R}.$$

Since M is compact, the limit exists and does not depend on the point $\tilde{x}$ (see [22]). The volume entropy is a way of describing the asymptotic behavior of the volumes of balls in the universal cover of a given Riemannian manifold.

An example of a manifold with "complicated topology" is a manifold of hyperbolic type, i.e., a manifold which admits a hyperbolic Riemannian metric. Let (M^n, hyp) be a closed hyperbolic manifold. The volume of a ball in the hyperbolic space $\mathbb{H}^n$, i.e., the universal cover of (M^n, hyp), is independent of the center of the ball. Thus $V_{\mathbb{H}^n}(R)$ is just the volume of any ball of radius R in the hyperbolic n-space, which can be explicitly calculated. In particular, when $n = 2$, for every $R > 0$ we have

$$V_{\mathbb{H}^2}(R) = 2\pi(\cosh(R) - 1). \tag{1.1.1}$$

So there exists a constant c such that

$$V_{\mathbb{H}^2}(R) \sim ce^R,$$

when R goes to infinity.

Now let h be another metric on M with $\mathrm{Vol}(M, h) \leq \mathrm{Vol}(M, hyp)$. Does the balls in $(\widetilde{M}, \widetilde{h})$ also grow exponentially like in the hyperbolic case ? There exist two fundamental theorems in this direction. The first theorem is due to G. Besson, G. Courtois, S. Gallot [2] and also to A. Katok [18] for the dimension $n = 2$. The authors proved that if M is a closed connected Riemannian manifold that carries a rank one locally symmetric metric h_0, then for every Riemannian metric h such that $\mathrm{Vol}(M, h) = \mathrm{Vol}(M, h_0)$, the inequality $\mathrm{Ent}(M, h) \geq \mathrm{Ent}(M, h_0)$ holds. In our language their theorem can be expressed as follows.

Theorem 1.1.1 (see [2], [18])**.** *Let (M^n, hyp) be a closed hyperbolic manifold, and let h be another metric on M with* $\mathrm{Vol}(M, h) < \mathrm{Vol}(M, hyp)$. *Then there is some constant R_0 (depending on the metric h) such that for every radius $R > R_0$, the following inequality holds :*

$$V_{(\widetilde{M}, \widetilde{h})}(R) > V_{\mathbb{H}^n}(R).$$

It would be interesting to know the value of R_0 in Theorem 1.1.1 since we are looking for a lower bound on the function $V_{(\widetilde{M}, \widetilde{h})}$ for every $R \geq 0$.
The second fundamental theorem can be seen as a first step toward estimating R_0 but with a stronger hypothesis.

Theorem 1.1.2 (Guth, [16])**.** *For every dimension n, there is a number $\delta(n) > 0$ such that if (M^n, hyp) is a closed hyperbolic n-manifold and h is another metric on M with* $\mathrm{Vol}(M, h) < \delta(n)\,\mathrm{Vol}(M, hyp)$, *then the following inequality holds*

$$V_{(\widetilde{M}, \widetilde{h})}(1) > V_{\mathbb{H}^n}(1).$$

The method presented in [16] can be modified to give a similar estimate for balls of radius R. For each R, there is a constant $\delta(n, R) > 0$ such that if $\mathrm{Vol}(M, g) < \delta(n, R)\,\mathrm{Vol}(M, hyp)$ then $V_{(\tilde{M}, \tilde{g})}(R) > V_{\mathbb{H}^n}(R)$. As R goes to infinity, the constant $\delta(n, R)$ falls off exponentially or faster so this method become less effective, whereas the methods in [2] are only effective asymptotically for very large R. This led L. Guth to ask if we can get a uniform estimate for $R \geq 1$. In other words, the question is : does there exist a positive constant $\delta(n)$ such that $\mathrm{Vol}(M, g) < \delta(n)\,\mathrm{Vol}(M, hyp)$ implies $V_{(\tilde{M}, \tilde{g})}(R) > V_{\mathbb{H}^n}(R)$ for all $R \geq 1$?

Here we positively answer Guth's question for the dimension $n = 2$.

Theorem I. *There exists a positive constant δ such that if (M, hyp) is a closed hyperbolic surface and h is another metric on M with $\operatorname{Area}(M,h) \leq \delta \operatorname{Area}(M, hyp)$, then for any radius $R \geq 1$,*

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq V_{\mathbb{H}^2}(R).$$

Our Theorem I will be deduced from the following more general theorem.

Theorem II. *There exists two small positive constants δ and c such that if (M, hyp) is a closed hyperbolic surface and h is another metric on M with $\operatorname{Area}(M,h) \leq \delta \operatorname{Area}(M, hyp)$, then for any radius $R \geq 0$,*

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq V_{\mathbb{H}^2}(cR).$$

We can extend the notion of entropy from Riemannian manifolds to metric graphs. Let (Γ, h) be a metric graph and denote by $(\widetilde{\Gamma}, \widetilde{h})$ its universal cover. Fix a point v of Γ and a lift $\tilde{v}$ of this point in $\tilde{\Gamma}$. The volume entropy of (Γ, d) is defined as

$$\operatorname{Ent}(\Gamma, h) = \lim_{R\to\infty} \frac{\log(\operatorname{length}(B_{\widetilde{h}}(\tilde{v}, R)))}{R}.$$

Since Γ is compact, the limit exists and does not depend on the point $\tilde{v}$ (see [22]).

Definition 1.1.1. *Let (Γ, h) be a metric graph and denote by $(\widetilde{\Gamma}, \widetilde{h})$ its universal cover. We define the function*

$$V'(R) := \sup_{\tilde{v}\in\widetilde{\Gamma}} \operatorname{length}(B_{\widetilde{h}}(\tilde{v}, R)),$$

where $B_{\widetilde{h}}(\tilde{v}, R)$ is a ball of radius R centered at the point $\tilde{v}$ of $\widetilde{\Gamma}$.

A regular graph is the analog of a Riemannian manifold carrying a locally symmetric metric. For every positive integer $b \geq 2$, we denote by Γ_b a connected trivalent graph of first Betti number b and by h_b the metric on Γ_b for which all the edges have length 1. In [19] (see also [20]), the authors proved a theorem for graphs analog to the G. Besson, G. Courtois and S. Gallot theorem for manifolds. They showed that for every integer $b \geq 2$ and every connected metric graph (Γ, h) of first Betti number b such that $\operatorname{length}(\Gamma, h) = \operatorname{length}(\Gamma_b, h_b)$, we have $\operatorname{Ent}(\Gamma, h) \geq \operatorname{Ent}(\Gamma_b, h_b)$. In our language, their theorem can be stated as follows.

Theorem 1.1.3 ([19],[20])**.** *Let (Γ, h) be a connected metric graph of first Betti number $b \geq 2$ Such that $\operatorname{length}(\Gamma, h) < \operatorname{length}(\Gamma_b, h_b)$. Then there exists some constant R'_0 (depending on the metric h) such that for every radius $R > R'_0$ the following inequality holds*

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

In view of Theorems 1.1.2 and 1.1.3, one can ask the following question : does there exist a universal constant $c > 0$ such that if $\text{length}(\Gamma, h) < c\,\text{length}(\Gamma_b, h_b)$, then for all $R \geq 0$

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R)?$$

We give a partial answer to this question.

Theorem III. *Fix $\lambda \in \left]0, \frac{1}{6}\right]$. Let (Γ, h) be a connected metric graph of first Betti number $b \geq 2$ such that*

$$\text{length}(\Gamma, h) \leq \lambda\,\text{length}(\Gamma_b, h_b).$$

Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have

$$\text{length}\, B_{\widetilde{h}}(\tilde{u}, R) \geq (1 - 3\lambda) V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq (1 - 3\lambda) V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

We sketch an outline of the main idea of the proof of Theorem II. Fix $R \geq 0$ and denote by g the genus of M. First, we show that we can suppose that the systole $\text{sys}(M, h)$ of (M, h) is at least $max\{2R, 1/2\}$. This lower bound on the systole and the upper bound on the area of the surface in terms of the genus permit us to show the existence of an embedded minimal graph Γ in M which captures the topology of the surface (*cf.* Definition 1.5.1 and Definition 1.5.3) and satisfies the hypothesis of Theorem III. Therefore, there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for all radii $r \in (0, R)$, the length of the ball $B_{\widetilde{\Gamma}}(r)$ in $\widetilde{\Gamma}$ centered at $\tilde{u}$ and of radius r is large. Since $R \leq \frac{1}{2}\,\text{sys}(\Gamma, h)$, the length of the projection $B_\Gamma(r)$ of $B_{\widetilde{\Gamma}}(r)$ in Γ is also large. Let $B_M(r)$ be the ball of radius r in M with the same center as $B_\Gamma(r)$. For all radii $r \leq R$, the boundary of $B_M(r)$ is at least as long as the graph $\Gamma \cap B_M(r)$, for otherwise we could construct another graph Γ' which captures the topology of M and is shorter than Γ. This would contradict the minimality of Γ. Since the graph $\Gamma \cap B_M(r)$ contains $B_\Gamma(r)$, we derive that the length of $\partial B_M(r)$ is large. By the coarea formula, we conclude that the area of $B_M(R)$ is also large.

This paper is organized as follows. In Section 1.2, we recall the basic material of graphs we need in this paper. In Section 1.3, we prove a special case of Theorem III. In Section 1.4, we prove Theorem III in the general case. In Section 1.5, we show the existence of graphs that captures the topology of closed orientable Riemannian surfaces. In Section 1.6, we extend the notion of the height function originally defined by Gromov for surfaces, then we show a relation between the height and the area of balls. In Section 1.7, we establish the existence of ε-regular metrics. In Section 1.8, we define short minimal graphs on surfaces that capture the topology and we study their properties. At the end of this section, we show how to control their length in terms of the genus of the surface. In Section 1.9, we give the proof of the main theorems I and II.

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1.2 Preliminaries

By a graph Γ we mean a finite one-dimensional CW-complex (multiple edges and loops are allowed). It is also useful to see Γ as a pair of sets (V, E) where V is a set of vertices and E the set of edges, which are 2-element subsets of V. Two vertices of a graph are called *adjacent* if there is an edge linking them. An edge and a vertex are called *incident* if the vertex is an endpoint of the edge. The *degree* (also known as valence) of a vertex v, denoted by $\deg(v)$, is the number of edges incident to it, where the loops are counted twice. We say that a graph Γ is *k-regular* if the degree of any vertex is k. In particular, a 3-regular graph is called *trivalent*. The minimal degree of a graph Γ is the minimum of the degrees of the vertices. It will be denoted by $\mathrm{Mindeg}(\Gamma)$. A graph Γ with $\mathrm{Mindeg}(\Gamma) \geq 3$ is called at least trivalent. For a graph Γ, we always denote by $E(\Gamma)$ the set of its edges and by $V(\Gamma)$ the set of its vertices. The first Betti number of a graph Γ can be computed as follows :

$$b(\Gamma) = e - v + n, \tag{1.2.1}$$

where e, v and n are respectively the number of edges, vertices and connected components of Γ.

The degree sum formula states that, given a graph Γ, we have that

$$\sum_v \deg(v) = 2e, \tag{1.2.2}$$

where the summation is over all vertices v of Γ.

For an at least trivalent connected graph Γ with first Betti number b, we have that $2e \geq 3v$ by (2.2). Combined with (2.1), we get $e \leq 3b - 3$. That means that the number of edges of Γ is bounded in terms of its first Betti number b. Also it is not hard to see from (2.1) and (2.2) that every connected graph of first Betti number $b \geq 2$ has at least one vertex of degree at least 3.

Let Γ be a connected graph, v_0 and v_1 be two vertices of Γ. A path P from v_0 to v_1 is a sequence of directed edges that links v_0 to v_1. The vertex v_0 is called the start point of P and v_1 the endpoint. If $v_0 = v_1$ then P is said to be closed, otherwise P is open. A simple path is a path with no self intersections. A simple closed path is often called a cycle.

A metric graph (Γ, h) is a graph endowed with a metric h such that (Γ, h) is a length space. The length of a subgraph of Γ is its one-dimensional Hausdorff measure. For more details on graphs we refer the reader to [8].

Throughout this paper if R is a real number then $[R]$ is the integral part of R.

For the connected trivalent metric graph (Γ_b, h_b) of first Betti number $b \geq 2$ where edges are of unit length, the following holds :

– $$\text{length}(\Gamma_b, h_b) = 3b - 3. \tag{1.2.3}$$

– The universal cover $\widetilde{\Gamma}_b$ is isometric to the trivalent infinite tree. In particular, $\widetilde{\Gamma}_b$ is independent of b. So for every $b' \geq 2$ we have

$$V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R) = V'_{(\widetilde{\Gamma}_{b'}, \widetilde{h}_{b'})}(R).$$

– For every $R \geq 0$ and every vertex $\tilde{v}$ of $(\widetilde{\Gamma}_b, \widetilde{h}_b)$, we have

$$\begin{aligned} \text{length}(B_{\widetilde{h}_b}(\tilde{v}, R)) &= 3\sum_{n=0}^{[R]-1} 2^n + 3(R - [R])2^{[R]} \\ &= 3(2^{[R]} - 1) + 3(R - [R])2^{[R]} \\ &\geq \sinh(R \ln 2). \end{aligned} \tag{1.2.4}$$

Therefore, $V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R) \geq \sinh(R \ln 2)$.
In particular, one should notice that the volume of the ball $B_{\widetilde{h}_b}(\tilde{v}, R)$ is independent from the vertex $\tilde{v}$ and from the first Betti number b. It only depends on R.

1.3 Baby theorem III

In this section, we prove Theorem III with an additional bound on the lengths of the edges of Γ and on the minimal degree of Γ (*cf.* Section 1.2).

Proposition 1.3.1. *Let c and C' be two positive constants with $c \leq C'$. Let (Γ, h) be a connected, at least trivalent metric graph of first Betti number $b \geq 2$ such that the edges of Γ are of length at most c. Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have*

$$\text{length}\, B_{\widetilde{h}}(\tilde{u}, (C' + c)R) \geq C' V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}, \widetilde{h})}((C' + c)R) \geq C' V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

Proof. Let $\mathcal{T}$ be a connected trivalent infinite subgraph of $\widetilde{\Gamma}$. We will construct a connected trivalent infinite subgraph $\mathcal{T}'$ of $\mathcal{T}$ for which there exists an homeomorphism f : $\widetilde{\Gamma}_b \to \mathcal{T}'$ that satisfies the following :
For every pair of vertices x, y of $\widetilde{\Gamma}_b$, we have

$$C'd(x, y) \leq d(f(x), f(y)) \leq (C' + c)d(x, y). \tag{1.3.1}$$

For the sake of clarification, we will do this construction step by step.

Step 1 : Start by fixing a vertex v_0 in $\mathcal{T}$. Let e_{1v_0} be one of the three edges of $\mathcal{T}$ incident to v_0 and denote by v_1 its second endpoint. Again let e_{1v_1} be one of the other two edges of $\mathcal{T}$ incident to v_1 and denote by v_2 its second endpoint. The path $e_{1v_0}e_{1v_1}$ is simple and open. We continue doing this by induction and we denote by v_k the first vertex where the length of the path $e_{1v_0}...e_{1v_k}$ is at least C'. The graph $\mathcal{T}$ contains no nontrivial cycles since it is a tree. That means that the path $p_1 = e_{1v_0}...e_{1v_k}$ is simple and open. Furthermore, the length of p_1 is between C' and $C'+c$. Now take the second edge e_{2v_0} of $\mathcal{T}$ incident to v_0 and restart the process of Step 1. This give us another simple open path p_2. Again, since $\mathcal{T}$ contains no nontrivial cycles the intersection $p_1 \cap p_2$ is the vertex v_0. Also restart the process with the third edge of $\mathcal{T}$ incident to v_0 to get the third path p_3.

Step 2 : The tree $X = p_1 \cup p_2 \cup p_3$ has three leaves. For each leaf x_i of X there are two edges of $\mathcal{T}$ incident to it other than the edge that is already in X. So by restarting the process of Step 1, we construct two paths of length at least C' with start point x_i. By induction, we keep doing what we did before to finally get the subgraph $\mathcal{T}'$. In what follows each path p_i of the subgraph $\mathcal{T}'$ will be seen as an edge of the same length of p_i. That means $\mathcal{T}'$ can be seen as a connected infinite trivalent subgraph of $\mathcal{T}$ where the length of any edge of $\mathcal{T}'$ is between C' and $C'+c$. The graphs $\widetilde{\Gamma}_b$ and $\mathcal{T}'$ are two infinite trivalent trees so there exists an homeomorphism $f : \widetilde{\Gamma}_b \to \mathcal{T}'$ that sends every edge of $\widetilde{\Gamma}_b$ to an edge of $\mathcal{T}'$.

Now we prove that the map f satisfies (3.1). Without loss of generality, we will prove our claim when x and y are the endpoints of the same edge e_{xy} in $\widetilde{\Gamma}_b$, that is, $d(x,y) = 1$. By construction of the map f, the length of the image of an edge of $\widetilde{\Gamma}_b$ is between C' and $C'+c$. So clearly

$$C'd(x,y) \leq d(f(x), f(y)) \leq (C'+c)d(x,y).$$

Now let $\tilde{u}$ be a vertex of $\mathcal{T}'$ and denote by w its inverse image in $\widetilde{\Gamma}_b$. By (3.1), we have

$$\begin{aligned} C' \operatorname{length} B_{\widetilde{h}_b}(w,R) &\leq \operatorname{length}(f(B_{\widetilde{h}_b}(w,R))) \\ &\leq \operatorname{length}(B_{\widetilde{h}}(\tilde{u},(C'+c)R)), \end{aligned}$$

Hence the proposition. □

1.4 Proof of theorem III

In this section, we prove Theorem III. As a preliminary, let us examine how the function V' changes with scaling. Let (Γ, h) be a metric graph and $h' = \mu h$ with $\mu > 0$ then

- $\operatorname{length}(\Gamma, h') = \mu \operatorname{length}(\Gamma, h)$;
- $V'_{(\widetilde{\Gamma},\widetilde{h}')}(\mu R) = \mu V'_{(\widetilde{\Gamma},\widetilde{h})}(R)$.

Definition 1.4.1. *Let Γ be a connected metric graph of first Betti number at least two. If v is a vertex of Γ of degree two then by the sentence* "ignore the vertex v" *we mean delete the two edges e_1 and e_2 of Γ incident to v and replace them by an edge of* length $\operatorname{length}(e_1) + \operatorname{length}(e_2)$ *that links the other two vertices of e_1 and e_2.*

Lemma 1.4.1. *Let (Γ, h) be a connected metric graph of first Betti number $b \geq 2$. There exists a metric graph (Γ', h') with first Betti number $b' = b$ that satisfies the following.*

- *Γ' is at least trivalent ;*
- *$\mathrm{length}(\Gamma', h') \leq \mathrm{length}(\Gamma, h)$;*
- *For all $R \geq 0$,*

$$V'_{(\widetilde{\Gamma}', \widetilde{h}')}(R) \leq V'_{(\widetilde{\Gamma}, \widetilde{h})}(R).$$

Proof. First we remove every vertex of Γ of degree one along with the edge incident to it and denote by Γ_1 the resulting connected graph. We apply the same process to Γ_1. That means we remove every vertex of Γ_1 of degree one along with the edge incident to it and we denote by Γ_2 the resulting connected graph . By induction, let Γ_k be the last connected graph where no vertex of degree one left. The graph Γ_k is of first Betti number b and of length less or equal to the length of Γ. We keep denoting by h the restriction of the metric h to Γ_k. The universal cover $\widetilde{\Gamma}_k$ is isometrically embedded into $\widetilde{\Gamma}$ so

$$V'_{(\widetilde{\Gamma}, \widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_k, \widetilde{h})}(R).$$

Second, we ignore every vertex of Γ_k of degree two (*cf.* Definiton 1.4.1). The resulting graph Γ' is connected of first Betti number b and of the same length as Γ_k. The universal cover $\widetilde{\Gamma}'$ agrees with $\widetilde{\Gamma}_k$ so

$$V'_{(\widetilde{\Gamma}', \widetilde{h})}(R) = V'_{(\widetilde{\Gamma}_k, \widetilde{h})}(R).$$

□

In order to prove Theorem III, it is convenient here to reformulate it. Given $\lambda \in \left]0, \frac{1}{6}\right]$, let c and C' be two positive constants such that $c \leq C'$ and $\lambda = \frac{c}{3(C'+c)}$. So a reformulated version of Theorem III is the following.

Theorem 1.4.1. *Let (Γ, h) be a connected metric graph of first Betti number $b \geq 2$. Let C' and c be two positive constants with $c \leq C'$. Suppose that*

$$\mathrm{length}(\Gamma, h) \leq \frac{c}{3(C'+c)} \, \mathrm{length}(\Gamma_b, h_b).$$

Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have

$$\mathrm{length}\, B_{\widetilde{h}}(\tilde{u}, R) \geq \frac{C'}{C'+c} V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}, \widetilde{h})}(R) \geq \frac{C'}{C'+c} V'_{(\widetilde{\Gamma}_b, \widetilde{h}_b)}(R).$$

Proof. By scaling, we will prove the following. Suppose that

$$\mathrm{length}(\Gamma, h) \leq \frac{c}{3} \, \mathrm{length}(\Gamma_b, h_b) = c(b-1).$$

Then there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that for any $R \geq 0$, we have

$$\operatorname{length} B_{\widetilde{h}}(\tilde{u},(C'+c)R) \geq C' V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma},\widetilde{h})}((C'+c)R) \geq C' V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

First notice that by Lemma 1.4.1, we can suppose that Γ is at least trivalent. We proceed by induction on the first Betti number of Γ. For $b = 2$, we have

$$\max_{e \in E} \operatorname{length}(e) < \operatorname{length}(\Gamma, h) \leq c(2-1) = c.$$

By Proposition 1.3.1, the result follows in this case.
Suppose the result holds for $b = n$ and let us show that it also for $b = n+1$. Let (Γ, h) be a connected metric graph of first Betti number $b = n+1$. If Γ contains no edge of length greater than c then the result follows from Proposition 1.3.1. Thus we suppose the opposite here and remove an edge w of Γ of length greater than c. There are two cases to consider.

Case 1 : The edge w is non-separating in Γ. In this case, the resulting graph Γ' is connected and of first Betti number $b' = n$. Furthermore, we have

$$\operatorname{length}(\Gamma') \leq \operatorname{length}(\Gamma) - c \leq c(b'-1).$$

The universal cover $\widetilde{\Gamma}'$ is isometrically embedded into $\widetilde{\Gamma}$. So for every vertex $\tilde{v}$ in $\widetilde{\Gamma}'$ and every $R > 0$, we have

$$\operatorname{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{v},R)) \geq \operatorname{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{v},R)).$$

In particular, we have

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}',\widetilde{h})}(R).$$

On the other hand, by the hypothesis of the induction, we know that there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}'$ such that

$$\operatorname{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{u},R)) \geq V'_{(\widetilde{\Gamma}_n,\widetilde{h}_n)}(R) = V'_{(\widetilde{\Gamma}_{n+1},\widetilde{h}_{n+1})}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}',\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_{n+1},\widetilde{h}_{n+1})}(R).$$

This finishes the proof in this case.

Case 2 : The edge w is separating in Γ. Thus, it splits the graph Γ into two connected graphs Γ' and Γ'' of first Betti number b' and b''. We claim that $\operatorname{length}(\Gamma') \leq c(b'-1)$ or $\operatorname{length}(\Gamma'') \leq c(b''-1)$. Indeed, suppose the opposite then

$$\operatorname{length}(\Gamma') + \operatorname{length}(\Gamma'') > c(b-2).$$

On the other hand we have

$$\mathrm{length}(\Gamma') + \mathrm{length}(\Gamma'') + c < \mathrm{length}(\Gamma) \leq c(b-1).$$

Hence a contradiction. So the claim is proved.
Without loss of generality, suppose that Γ' satisfies $\mathrm{length}(\Gamma') \leq c(b'-1)$. Clearly $b' \geq 2$, otherwise the length of Γ' would vanish. By induction, we now there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}'$ such that

$$\mathrm{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{u}, R)) \geq V'_{(\widetilde{\Gamma}_{b'},\widetilde{h}_{b'})}(R) = V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

In particular, we have

$$V'_{(\widetilde{\Gamma}',\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}_b,\widetilde{h}_b)}(R).$$

Recall that the universal cover $\widetilde{\Gamma}'$ is isometrically embedded into $\widetilde{\Gamma}$. So for every vertex $\tilde{v}$ in $\widetilde{\Gamma}'$ and every $R > 0$, we have

$$\mathrm{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{v}, R)) \geq \mathrm{length}(B_{(\widetilde{\Gamma}',\widetilde{h})}(\tilde{v}, R)).$$

In particular, we have

$$V'_{(\widetilde{\Gamma},\widetilde{h})}(R) \geq V'_{(\widetilde{\Gamma}',\widetilde{h})}(R).$$

This finishes the proof in this case too.

□

1.5 Capturing the topology of surfaces

In this section, we show that on every closed orientable Riemannian surface M there exist an embedded graph that captures its topology.

Definition 1.5.1. *Let (M, h) be a closed Riemannian surface of genus g. The image in M of an abstract graph by an embedding will be refered to as a graph in M. The metric h on M naturally induces a metric on a graph Γ in M. Despite the risk of confusion, we will also denote by h such a metric on Γ.*

We say that a graph Γ in M captures the topology *of M if the map induced by the inclusion $i_* : H_1(\Gamma, \mathbb{R}) \to H_1(M, \mathbb{R})$ is an epimorphism.*

Lemma 1.5.1. *Let (M, h) be a closed orientable Riemannian surface. Let Γ be a connected graph in M that captures its topology and denote by $i : \Gamma \to M$ the inclusion map. Then there exists a connected subgraph Γ' of Γ such that the map i_* restricted to Γ' is an isomorphism. In particular the first Betti number of Γ' is $2g$.*

Proof. Let Γ' be a connected subgraph of Γ with minimal number of edges such that the restriction of i to Γ' still induces an epimorphism in real homology. Let α be a cycle of Γ' representing a nontrivial element of the kernel of i_*. Remove an edge e from α. The resulting graph Γ'' has fewer edges than Γ'. Let β be a cycle of Γ'. If e does not lie in β then the cycle $\gamma = \beta$ lies in Γ''. Otherwise, adding a suitable real multiple of α to β

yields a new cycle γ lying in Γ''. In both cases, the cycle γ of Γ'' is sent to the same homology class as β by i_*. Thus, the restriction of i to Γ'' still induces an epimorphism in the real homology, which is absurd by definition of Γ'. $\square$

In what follows a graph Γ in a Riemannian manifold (M, h) is automatically equipped with the metric h induced by the metric of M. So the length of Γ is its one-dimensional Hausdorff measure associated to the metric h.

Definition 1.5.2. *Let (M, h) be a closed orientable Riemannian surface. We define*

$$L(M, h) := \inf_{\Gamma} \operatorname{length}(\Gamma),$$

where the infimum is taken over all graphs Γ in M that capture its topology.

Lemma 1.5.2. *Let (M, h) be a closed orientable Riemannian surface of genus g. Then there exists a graph Γ in M that captures its topology with*

$$\operatorname{length}(\Gamma) = L(M, h).$$

Proof. By Lemma 1.5.1, we only need to consider the set of graphs in M that captures its topology with first Betti number $2g$ and such that i_* is an isomorphism. Furthermore, we only need to consider graphs that are at least trivalent. Indeed, delete every vertex of Γ of degree one along with the edge incident to it. Denote by Γ_1 the resulting connected graph and apply to Γ_1 the same process. That means we delete every vertex of Γ_1 of degree one along with the edge incident to it and we denote by Γ_2 the resulting connected graph . By induction, let Γ_k be the last connected graph with no vertex of degree one. We then ignore all vertices of Γ_k of degree two (*cf.* Definition 1.4.1). Replacing every edge of Γ_k by a minimal representative of its fixed-endpoint homotopy class gives rise to a geodesic graph Γ'. By construction the connected geodesic graph Γ' is at least trivalent and of first Betti number $2g$. Thus, its number of edges is bounded in terms of g, *cf.* Section 1.2. Now the space of connected geodesic graphs of M capturing its topology with bounded length and a bounded number of edges is compact. The result follows. $\square$

Definition 1.5.3. *Let (M, h) be a closed orientable Riemannian surface. If Γ is a graph that captures the topology of M with* $\operatorname{length}(\Gamma) = L(M, h)$*, then Γ is called a minimal graph in M.*

1.6 Height function and area of balls.

In this section, we first recall the definition of the *height* function on surfaces defined by Gromov in [10] along with its relation to the area of balls. Then we extend this notion to make it suit our problem.

Let M be a closed Riemannian manifold. The systole at a point x in M, denoted by $\operatorname{sys}(M, x)$, is the length of the shortest non-contractible loop based at x. The systole of M, denoted by $\operatorname{sys}(M)$, is the length of the shortest non-contractible loop in M.

Definition 1.6.1. *Let (M, h) be a closed Riemannian surface and γ be a non-contractible loop in M. We define the tension of γ as follows.*

$$\text{tens}(\gamma) = \text{length}(\gamma) - \inf_{\beta \sim \gamma}(\text{length}(\beta)),$$

where the infimum is taken over all closed curves β freely homotopic to γ.

We also define the height function H' on M as follows

$$H'(x) = \inf_{\gamma}(\text{tens}(\gamma)),$$

where the infimum is taken over all non-contractible closed curves γ passing through x.

Proposition 1.6.1 (Gromov, [10] Proposition 5.1.B)**.** *Let (M, h) be a complete Riemannian surface and $x \in M$. Then*

$$\text{Area}\, B(x, R) \geq \frac{1}{2}(2R - H'(x))^2,$$

for every R in the interval $[\frac{1}{2}H'(x), \frac{1}{2}\operatorname{sys}(M, x)]$.

Definition 1.6.2. *Let (M, h) be a closed orientable Riemannian surface. For $x \in M$, we define*

$$L(M, x) := \inf_{\Gamma_x} \text{length}(\Gamma_x),$$

where the infimum is taken over all graphs Γ_x in M that capture its topology and pass through x.

We also define the function H'' on M as follows.

$$H''(x) := L(M, x) - L(M, h).$$

Finally we define the function H on M as

$$H(x) := min(H'(x), H''(x)),$$

where H' is defined in Definition 1.6.1.

Definition 1.6.3. *If B is a ball in some closed Riemannian surface M with some contractible boundary components, we fill in every such component of ∂B by an open 2-cell in M and denote by B^+ the union of B with these cells.*

Proposition 1.6.2. *Let (M, h) be a closed Riemannian surface of genus $g \geq 1$ and $x \in M$ with $H(x) < \frac{1}{2}\operatorname{sys}(M, x)$. Then the area of the ball $B(x, R)$ satisfies the inequality*

$$\text{Area}\;\; B(x, R) \geq \frac{1}{2}(R - H(x))^2,$$

for every R in the interval $]H(x), \frac{1}{2}\operatorname{sys}(M, x)[$.

Proof. We suppose that $H(x) = H''(x)$ here, since the other case follows from Proposition 1.6.1. Let $r \in]H(x), \frac{1}{2}\operatorname{sys}(M,x)[$. Notice that since $r < \frac{1}{2}\operatorname{sys}(M,x)$ the ball $B = B(x,r)$ is contractible in M, and so the set $B^+ = B^+(x,r)$ is a topological disk. Let ε be a fixed small positive constant such that $H''(x) + \varepsilon < r$. Fix $\varepsilon' \in (0,\varepsilon)$. Let Γ_x be a graph in M that captures its topology and passes through x of length at most $L(M,x) + \varepsilon'$. Without loss of generality, we claim that we can always suppose that $\Gamma_x \cap B^+(x,r)$ is a tree such that x is the only possible vertex of degree one. Indeed, we delete an edge from each loop of $\Gamma_x \cap B^+(x,r)$. This defines a new graph Γ'. Then we delete every vertex of Γ' of degree one other than the vertex x along with the edge incident to it and we denote by Γ_1 the resulting connected graph. Restart the process. That means we delete every vertex of Γ_1 of degree one other than the vertex x along with the edge incident to it and we denote by Γ_2 the resulting connected graph. By induction, let Γ_k be the last connected subgraph where the only possible vertex of degree one is x. Clearly Γ_k passes through x, captures the topology of M and is of length at most $L(M,x)+\varepsilon'$. So the claim is proved.

Now we claim that either x is of degree at least two or there is at least a vertex of $\Gamma_x \cap B^+$ of degree at least three. Indeed, suppose that x is of degree one and all the other vertices of $\Gamma_x \cap B^+$ are of degree two. Then $\Gamma_x \cap B^+$ is just a piecewise curve that passes through x and hits ∂B^+ at one point, so its length is greater or equal to r. Thus

$$\operatorname{length}(\Gamma_x) \geq L(M,h) + r.$$

In particular, we have

$$L(M,h) + r \leq L(M,x) + \varepsilon' \leq L(M,x) + \varepsilon.$$

That means

$$r \leq H''(x) + \varepsilon,$$

which is a contradiction.

In both cases above, the graph Γ_x hits the boundary of B^+ in at least two points. Let C be a minimal arc of ∂B^+ that connects the points of $\Gamma_x \cap \partial B^+$. Consider the graph Γ' defined as $(\Gamma_x \setminus (\Gamma_x \cap B^+)) \cup C$. It is clear that Γ' is a connected graph in M that captures its topology, since B^+ is contractible in M. Thus

$$\operatorname{length}(\Gamma') \geq L(M,h).$$

On the other hand, the length of $\Gamma_x \cap B^+$ is at least r. This means that

$$\operatorname{length}(\Gamma_x) \geq \operatorname{length}(\Gamma') + r - \operatorname{length}(C).$$

So

$$L(M,x) + \varepsilon' \geq L(M,h) + r - \operatorname{length}(C).$$

We conclude that for every small positive constant ε', we have

$$H''(x) \geq r - \operatorname{length}(C) - \varepsilon'.$$

Since the length of ∂B^+ is at least the length of the arc C, we have

$$\text{length}(\partial B^+) \geq r - H''(x).$$

By the coarea formula,

$$\begin{aligned} \text{Area}\, B(x,R) &\geq \int_0^R \text{length}(\partial B(x,r))dr \\ &\geq \int_{H''(x)}^R \text{length}(\partial B^+(x,r))dr \\ &= \frac{1}{2}(R - H''(x))^2. \end{aligned}$$

□

1.7 Existence of ε-regular metrics.

In this section, we define ε-regular metrics and prove their existence. The existence of ε-regular metrics will play a crucial role in controlling the length of minimal graphs on surfaces.

Definition 1.7.1. *Let (M,h) be a closed Riemannian surface. The metric h is called ε-regular if for all the points x in M, $H(x) \leq \varepsilon$.*

Lemma 1.7.1. *Let (M_0,h_0) be a closed Riemannian surface. Then for every $\varepsilon > 0$, there exists a Riemannian metric $\bar{h}$ on M_0 conformal to h_0 such that*

1. $\text{Area}(M_0,\bar{h}) \leq \text{Area}(M_0,h_0)$;
2. *$\bar{h}$ is ε-regular;*
3. $L(M_0,\bar{h}) = L(M_0,h_0)$ *;*
4. $\text{sys}(M_0,\bar{h}) = \text{sys}(M_0,h_0)$.

Proof. Take a point x_0 in M_0 where $H(x_0) = H_{h_0}(x_0) > \varepsilon$ and denote by M_1 the space M_0/B^+ obtained by collapsing $B^+ = B^+(x_0,\varepsilon)$ to x_0. Let $p_0 : M_0 \to M_1$ be the (non-expanding) canonical projection and h_1 be the metric induced by h on M_1. The Riemannian surface (M_1,h_1) clearly satisfies (1). If h_1 is not ε-regular, we apply the same process. By induction we construct a sequence of :

- balls $B_i^+ = B^+(x_i,\varepsilon)$ in M_i, where x_i is a point with $H_{h_i}(x_i) > \varepsilon$.
- Riemannian surfaces (M_i,h_i) where $M_i = M_{i-1}/B_{i-1}$ and h_i is the metric induced by h_{i-1} on M_i.
- non-expanding canonical projections $p_i : M_i \to M_{i+1}$.

This process stops when we get an ε-regular metric.

Now, we argue exactly as [24, Lemma 4.2] to prove that this process stops after finitely many steps. Let $B_1^i,\ldots,B_{N_i}^i$ be a maximal system of disjoint balls of radius $r/3$

in M_i. Since p_{i-1} is non-expanding, the preimage $p_{i-1}^{-1}(B_k^i)$ of B_k^i contains a ball of radius $r/3$ in M_{i-1}. Furthermore, the preimage $p_{i-1}^{-1}(x_i)$ of x_i contains a ball B_{i-1} of radius r in M_{i-1}. Thus, two balls of radius $r/3$ lie in the preimage of x_i under p_{i-1}. It is then possible to construct a system of $N_i + 1$ disjoint disks of radius $r/3$ in M_{i-1}. Thus, $N_{i-1} \geq N_i + 1$ where N_i is the maximal number of disjoint balls of radius $r/3$ in M_i. Therefore, the process stops after N steps with $N \leq N_0$. Denote by h_N the metric where this process stops. Clearly h_N satisfies (1) and (2). To see that h_N satisfies (3) and (4), let Γ be a minimal graph in M_0 and α be a systolic loop in M. For every point x in the ε-neighborhood N_Γ of Γ, we have $H(x) \leq \varepsilon$. Indeed, let c be a minimizing curve from Γ to x. The graph $\Gamma \cup c$ captures the topology of M_0 and passes through x. So $H''(x) \leq \text{length}(\Gamma \cup c) - L(M, h) \leq \varepsilon$. That means that the balls we collapsed through the whole process do not intersect Γ. Therefore, the metric h_N satisfies (3). A similar argument holds for α. So the metric h_N also satisfies (4).

□

1.8 Construction of short minimal graphs on surfaces

In this section, we combine Lemma 1.7.1 and the construction of [1, p. 46] to construct a minimal graph with controlled length on a given Riemannian surface.

Proposition 1.8.1. *Let (M, h) be a closed orientable Riemannian surface of genus $g \geq 2$. Suppose that*

- $\text{Area}(M, h) \leq \frac{1}{2^{12}}(2g - 1)$ *;*
- $\text{sys}(M, h) \geq \frac{1}{2}$.

Then

$$L(M, h) \leq \frac{1}{2}(2g - 1).$$

Proof. Fix $r_0 = \frac{1}{2^5}$. By Lemma 1.7.1 (choose ε small enough) and Proposition 1.6.2, there exists a conformal Riemannian metric $\bar{h}$ on M that satisfies

1. The area of every disk of $(M, \bar{h})$ of radius r_0 is at least $\frac{1}{4}{r_0}^2$;
2. $\text{Area}(M, \bar{h}) \leq \text{Area}(M, h)$;
3. $L(M, \bar{h}) = L(M, h)$;
4. $\text{sys}(M, \bar{h}) = \text{sys}(M, h)$
5. $\bar{h}$ is ε-regular.

So it is sufficient to prove that

$$L(M, \bar{h}) \leq \frac{1}{2}(2g - 1).$$

Let $\{B_i\}_{i \in I}$ be a maximal system of disjoint balls of radius r_0 in $(M, \bar{h})$. Since the area of each ball B_i is at least $\frac{1}{4}r_0^2$, then

$$\frac{1}{4}|I|r_0^2 \leq \text{Area}(M, \bar{h}),$$

that is,

$$|I| \leq 2^{12}\,\mathrm{Area}(M,\bar{h}). \tag{1.8.1}$$

As this system is maximal, the balls $2B_i$ of radius $2r_0$ with the same centers p_i as B_i cover M.

Let ε be a small positive constant that satisfies

$$4r_0 + 2\varepsilon < \frac{1}{4} \leq \frac{\mathrm{sys}(M,\bar{h})}{2},$$

and denote by $2B_i + \varepsilon$ the balls centered at p_i with radius $2r_0 + \varepsilon$. We construct an abstract graph Γ as follows. Let $\{w_i\}_{i\in I}$ be a set of vertices corresponding to $\{p_i\}_{i\in I}$. Two vertices w_i and $w_{i'}$ of Γ are linked by an edge if and only if the balls $2B_i + \varepsilon$ and $2B_{i'} + \varepsilon$ intersect each other. Define a metric on Γ such that the length of each edge is $\frac{1}{4}$ and let $\varphi : \Gamma \to M$ be the map that sends each edge of Γ with endpoints w_i and $w_{i'}$ to a minimizing geodesic joining p_i and $p_{i'}$. Since $\mathrm{dist}(p_i, p_{i'}) \leq 4r_0 + 2\varepsilon < \frac{1}{4}$, the map φ is distance nonincreasing.

Claim. The map $\varphi_\star : \pi_1(\Gamma) \to \pi_1(M)$ induced by φ between the fundamental groups is an epimorphism. In particular, it induces an epimorphism in real homology.

We argue exactly as [1, Lemma 2.10]. Consider a geodesic loop σ of M. Divide the loop σ into segments $\sigma_1, \ldots, \sigma_n$ of length at most ε. Denote by x_k and x_{k+1} the endpoints of σ_k with the convention $x_{n+1} = x_1$. Recall that the balls $2B_i$ cover the surface M. So every point x_k is at distance at most $2r_0$ from a point v_k among the centers p_i. Let β_k be the loop

$$\sigma_k \cup C_{x_{k+1}v_{k+1}} \cup C_{v_{k+1},v_k} \cup C_{v_k,x_k},$$

where C_{ab} denotes a minimizing geodesic joining a to b. We have that

$$\mathrm{length}(\beta_k) \leq 2\,(4r_0 + \varepsilon) < \mathrm{sys}(M,\bar{h}).$$

That means that the loops β_k are contractible. We conclude that the loop σ is homotopic to a piecewise geodesic loop $\sigma' = (v_1, \ldots, v_n)$.
The distance between the centers $v_k = p_{i_k}$ and $v_{k+1} = p_{i_{k+1}}$ is less than or equal to $4r_0 + \varepsilon$. So the vertices w_{i_k} and $w_{i_{k+1}}$ of Γ corresponding to the vertices p_{i_k} and $p_{i_{k+1}}$ are connected by an edge. The union of these edges forms a loop $(w_{i_1}, \ldots, w_{i_n})$ in Γ whose image by the map φ is σ'. Since σ' is homotopic to σ, the claim is proved.

Now we consider a connected subgraph Γ' of Γ with a minimal number of edges such that the restriction of φ to Γ' still induces an epimorphism in real homology.

We claim that the epimorphism $\varphi_* : H_1(\Gamma';\mathbb{R}) \to H_1(M;\mathbb{R})$ is an isomorphism. Indeed, if φ_* is not an isomorphism then arguing as in Proposition 1.5.1 we can remove at least one edge of Γ' such that φ_* is still an epimorphism, which is impossible by the definition of Γ'.
We denote by v, e, b and b' respectively the number of vertices of Γ, the number of edges

of Γ, the first Betti number of Γ and the first Betti number of Γ'. At least $b - b'$ edges were removed from Γ to obtain Γ'. As $b' = 2g$, we derive

$$\begin{aligned} \text{length}(\Gamma') &\leq \text{length}(\Gamma) - (b - b')\,\frac{1}{4} \\ &\leq (e - b + 2g)\,\frac{1}{4} \\ &\leq (v - 1 + 2g)\,\frac{1}{4}. \end{aligned} \tag{1.8.2}$$

Recall that $\text{Area}(M, \bar{h}) \leq \frac{1}{2^{12}}(2g - 1)$. So

$$v = |I| \leq 2g - 1.$$

Combining this with (6.2), we get

$$\text{length}(\Gamma') \leq \frac{1}{2}(2g - 1).$$

Since φ is distance non-increasing then

$$\text{length}(\varphi(\Gamma')) \leq \text{length}(\Gamma').$$

The image by φ of two edges of Γ' may intersect. If it is the case then the intersection point should be considered as a vertex of the graph $\varphi(\Gamma')$. Thus the set of vertices of $\varphi(\Gamma')$ may be bigger than the set of vertices of Γ'.

Finally let j be the inclusion map $j : \varphi(\Gamma') \hookrightarrow M$. Clearly the map $j_* : H_1(\varphi(\Gamma'); \mathbb{R}) \to H_1(M; \mathbb{R})$ is an epimorphism. So $\varphi(\Gamma')$ is a graph in M that captures its topology. Thus

$$L(M, \bar{h}) \leq \text{length}(\varphi(\Gamma')) \leq \frac{1}{2}(2g - 1).$$

□

1.9 Proofs of Theorem I and Theorem II.

In this section, we prove Theorem I and Theorem II. But before doing that we examine how the function V changes with scaling. Let (M^n, h) be a closed n-dimensional Riemannian manifold and $h' = \lambda^2 h$ with $\lambda > 0$ then
- $\text{Vol}(M, h') = \lambda^n \,\text{Vol}(M, hyp)$;
- $V_{(\widetilde{M}, \widetilde{h}')}(\lambda R) = \lambda^n V_{(\widetilde{M}, \widetilde{h})}(R)$.

The expression (1.1) of $V_{\mathbb{H}^2}$ immediately leads to the following lemma.

Lemma 1.9.1. *Let a be a positive constant. There exists a constant $c = c(a)$ such that for all $R \geq 0$,*

$$aV_{\mathbb{H}^2}(R) \geq V_{\mathbb{H}^2}(Rc).$$

In light of Lemma 1.9.1, the proof of Theorem II amounts to proving the following result.

Theorem 1.9.1. *Let* (M, hyp) *be a closed hyperbolic surface of genus* g *and* h *be another Riemannian metric on* M *with*

$$\text{Area}(M,h) \leq \frac{1}{2^{13}\pi}\,\text{Area}(M,hyp).$$

Then, for any radius $R \geq 0$,

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq \frac{1}{4\pi\ln 2} V_{\mathbb{H}^2}(R\ln 2).$$

In particular, there exists a constant c *such that*

$$V_{(\widetilde{M},\widetilde{h})}(R) \gtrsim c\,2^R,$$

when R *tends to infinity.*

Proof. Let $R > 0$. First, we consider the special case when M is oriented and

$$\text{sys}(M,h) \geq max\{2R, 1/2\}.$$

In this case,

$$V_{(M,h)}(R) = V_{(\widetilde{M},\widetilde{h})}(R).$$

Let Γ be a minimal graph which captures the topology of (M,h) (*cf.* Definition 1.5.3). Denote by $b = 2g$ the first Betti number of Γ. We have

$$\text{Area}(M,h) \leq \frac{1}{2^{13}\pi}\,\text{Area}(M,hyp) \leq \frac{1}{2^{12}}(2g-1).$$

So by Proposition 2.4.1 and the relation (2.3), we have

$$\text{length}(\Gamma) \leq \frac{1}{2}(b-1) = \frac{1}{6}\,\text{length}(\Gamma_b, h_b). \tag{1.9.1}$$

Let v be any vertex of Γ. Denote by $B(v,R)$ the ball in (M,h) centered at v with radius R. We claim that for all $r \in (0,R)$

$$\text{length}(\partial B^+(v,r)) \geq \text{length}(\Gamma \cap B^+(v,r)), \tag{1.9.2}$$

where $B^+(v,r)$ is defined in Definition 1.6.3.
We argue as in Proposition 1.6.2. Suppose the opposite and replace $\Gamma \cap B^+(v,r)$ by a minimal arc of $\partial B^+(v,r)$ that links the points of $\Gamma \cap \partial B^+(v,r)$. Since $B^+(v,r)$ is contractible, the new graph captures the topology of M and is shorter than Γ which contradicts the definition of Γ.

Let $B_{(\Gamma,h)}(v,r)$ be the ball centered at v of radius R in the metric graph (Γ,h). Since the ball $B_{(\Gamma,h)}(v,r)$ is contained in $\Gamma \cap B^+(v,r)$, we have

$$\text{length}(\Gamma \cap B^+(v,r)) \geq \text{length}(B_{(\Gamma,h)}(v,r)). \tag{1.9.3}$$

Let $\tilde{v}$ be a lift of v in $\widetilde{\Gamma}$. Since $\text{sys}(M,h) \leq \text{sys}(\Gamma,h)$, we have for $r \leq \frac{1}{2}\,\text{sys}(M,h)$

$$\text{length}(B_{(\Gamma,h)}(v,r)) = \text{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{v},r)). \tag{1.9.4}$$

By Theorem III (take $\lambda = \frac{1}{6}$) and the bound (9.1), there exists a vertex $\tilde{u}$ in $\widetilde{\Gamma}$ such that

$$\text{length}(B_{(\widetilde{\Gamma},\widetilde{h})}(\tilde{u},r)) \geq \frac{1}{2} V'_{(\widetilde{\Gamma}_{2g},\widetilde{h}_{2g})}(r).$$

Denote by u the image of $\tilde{u}$ by the covering map. By (9.2), (9.3), (9.4) and (2.4), we obtain

$$\begin{aligned} \text{length}(\partial B^+(u,r)) &\geq \frac{1}{2} V'_{(\widetilde{\Gamma}_{2g},\widetilde{h}_{2g})}(r) \\ &\geq \frac{1}{2}\sinh(r\ln 2). \end{aligned}$$

By the coarea formula,

$$\begin{aligned} \text{Area}(B(u,R)) &\geq \frac{1}{2}\int_0^R \sinh(r\ln 2)dr \\ &= \frac{1}{2\ln 2}(\cosh(R\ln 2) - 1). \\ &= \frac{1}{4\pi\ln 2} V_{\mathbb{H}^2}(R\ln 2). \end{aligned}$$

Next, we consider the general case with no restriction on the systole and the orientability of M. Since M admits a hyperbolic metric, the fundamental group of M is residually finite (see [21]). Therefore, we can choose a finite cover $(\bar{M},\bar{h})$ such that $\bar{M}$ is orientable and

$$\text{sys}(\bar{M},\bar{h}) \geq max\{2R, 1/2\}.$$

Let $\bar{hyp}$ be the pullback of the hyperbolic metric on M to $\bar{M}$.
Now, if the covering $\pi : \bar{M} \to M$ has degree d, then $\text{Area}(\bar{M},\bar{h}) = d\,\text{Area}(M,h)$ and $\text{Area}(\bar{M},\bar{hyp}) = d\,\text{Area}(M,hyp)$. So

$$\text{Area}(\bar{M},\bar{h}) \leq \frac{1}{2^{13}\pi}\,\text{Area}(\bar{M},\bar{hyp}).$$

Finally, since the universal cover of $(\bar{M},\bar{h})$ agrees with the universal cover of (M,h), we can conclude by the first case. □

Now we prove Theorem I.

Proof of Theorem I. Let (M, hyp) be a closed hyperbolic Riemannian surface of genus g. Let δ be a small positive constant and h another metric on M with $\mathrm{Area}(M,h) \leq \delta\,\mathrm{Area}(M, hyp)$. We will show that if we take δ small enough (independently from the metric h) then for any radius $R \geq 1$,

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq V_{\mathbb{H}^2}(R).$$

Indeed, let $h' = \lambda^2 h$ where λ is a positive constant such that

$$\mathrm{Area}(M,h') = \frac{1}{2^{13}\pi}\,\mathrm{Area}(M, hyp).$$

By Theorem 1.9.1, we have that for any radius $R \geq 0$,

$$V_{(\widetilde{M},\widetilde{h'})}(R) \geq \frac{1}{4\pi \ln 2} V_{\mathbb{H}^2}(R \ln 2).$$

Recall that

$$\mathrm{Area}(M,h') = \lambda^2\,\mathrm{Area}(M,h) \leq \lambda^2 \delta\,\mathrm{Area}(M, hyp).$$

So

$$\lambda^2 \geq \frac{1}{2^{13}\pi\delta}.$$

On the other hand, we have

$$V_{(\widetilde{M},\widetilde{h'})}(\lambda R) = \lambda^2 V_{(\widetilde{M},\widetilde{h})}(R).$$

So

$$V_{(\widetilde{M},\widetilde{h})}(R) \geq \frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2).$$

Now we choose λ large enough so that for all $R \geq 1$ we have

$$\frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2) \geq V_{\mathbb{H}^2}(R).$$

To see that such a λ exists notice that for $R \geq 1$ we have

$$\frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2) \geq \frac{1}{8\pi\lambda^2 \ln 2}(e^{\frac{\lambda \ln 2}{2}} e^{\frac{\lambda R \ln 2}{2}} - 2).$$

When λ tends to infinity, the number $\frac{1}{8\pi\lambda^2 \ln 2} e^{\frac{\lambda \ln 2}{2}}$ tends to infinity and so

$$\frac{1}{8\pi\lambda^2 \ln 2}(e^{\frac{\lambda \ln 2}{2}} e^{\frac{\lambda R \ln 2}{2}} - 2) \gg V_{\mathbb{H}^n}(R).$$

Recall that to get λ large enough it suffices to choose δ small enough.
Finally, we would like to point out that when R tends to zero we cannot find a λ such that

$$\frac{1}{4\pi\lambda^2 \ln 2} V_{\mathbb{H}^2}(\lambda R \ln 2) \geq V_{\mathbb{H}^2}(R).$$

$\square$

Chapitre 2

Courts lacets homotopiquement indépendants sur les surfaces

Short homotopically independent loops on surfaces

Abstract

In this paper, we are interested in short homologically and homotopically independent loops based at the same point on Riemannian surfaces and metric graphs.

First, we show that for every closed Riemannian surface of genus $g \geq 2$ and area normalized to g, there are at least $\lceil \log(2g)+1 \rceil$ homotopically independent loops based at the same point of length at most $C\log(g)$, where C is a universal constant. On the one hand, this result substantially improves Theorem 5.4.A of M. Gromov in [10]. On the other hand, it recaptures the result of S. Sabourau on the separating systole in [25] and refines his proof.

Second, we show that for any two integers $b \geq 2$ with $1 \leq n \leq b$, every connected metric graph Γ of first Betti number b and of length b contains at least n homologically independent loops based at the same point and of length at most $24(\log(b)+n)$. In particular, this result extends Bollobàs-Szemerédi-Thomason's $\log(b)$ bound on the homological systole to at least $\log(b)$ homologically independent loops based at the same point. Moreover, we give examples of graphs where this result is optimal.

2.1 Introduction

Short homotopically and homologically independent loops on surfaces have been of a great interest. Gromov proved in [10] and [13] that both $\mathrm{sys}(M)$, the systole, i.e., the shortest non-contractible loop, and $\mathrm{sys}_H(M)$, the homological systole, i.e., the shortest homologically nontrivial loop, of a closed Riemannian surface M of genus $g \geq 2$ with area normalized to $4\pi(g-1)$ are at most $\sim \log(g)$. In [1], F. Balacheff, S. Sabourau and H. Parlier found the maximal number of homologically independent loops of length at most $\sim \log(g)$. Their theorem goes as follows.

Theorem 2.1.1 ([1]). *Let $\eta : \mathbb{N} \to \mathbb{N}$ be a function such that*

$$\lambda := \sup_g \frac{\eta(g)}{g} < 1.$$

Then there exists a constant C_λ such that for every closed Riemannian surface M of genus g there are at least $\eta(g)$ homologically independent loops $\alpha_1, \ldots, \alpha_{\eta(g)}$ which satisfy

$$\mathrm{length}(\alpha_i) \leq C_\lambda \frac{\log(g+1)}{\sqrt{g}} \sqrt{\mathrm{Area}(M)},$$

for every $i \in \{1, \ldots, \eta(g)\}$.

Moreover, they constructed some hyperbolic surfaces where their bound is optimal.

For the applications we have in mind (see Section 2.2), it would be nice if the loops in Theorem 2.1.1 were based at the same point. Unfortunately, the following example shows that in general, we cannot even find two homologically independent loops based at the same point satisfying a $\log(g)$ bound. Indeed, let M be a closed hyperbolic surface of genus g. Consider a family of $g+1$ loops in M dividing the surface into two spheres with $g+1$ boundary components. Pinching these loops enough, we force (by the collar theorem) every loop of M homologically independent from this family to be arbitrary long. Still, we obtain some result in this direction when the systole is bounded from below, see Theorem 2.4.2.

This leads us to replace the notion of homologically independent loops by the notion of homotopically independent loops defined below.

Definition 2.1.1. *Let M be a closed Riemannian surface of genus at least one. A family of loops $(\alpha_1, ...\alpha_k)$ based at the same point v in M are said to be* homotopically independent *if the subgroup of $\pi_1(M, v)$ generated by $\alpha_1, \ldots, \alpha_k$ is free of rank k.*

Observe that k homologically independent loops based at the same point on a closed surface M of genus g are homotopically independent for $k < 2g$, see Theorem 2.4.1.

Now we ask the following question : for how many homotopically independent loops based at the same point does the $\log(g)$ bound hold ?

One might wonder or even doubt the benefit of finding short homotopically independent loops based at the same point. We show the benefits of such a choice in Section 2.2. To the author best knowledge, the only answer to the previous question is due to Gromov.

Theorem 2.1.2 ([10], 5.4.B)**.** *Let (M, h) be a closed Riemannian surface of genus $g \geq 2$ and of area normalized to g. For every $\alpha < 1$, there exist two homotopically independent loops γ_1 and γ_2 based at the same point in M such that*

$$sup(\text{length}(\gamma_1), \text{length}(\gamma_2)) \leq C_\alpha \, g^{1-\alpha},$$

where C_α is a positive constant that depends only on α.

Note that Theorem 2.1.2 does not hold for $\alpha = 1$. Indeed, P. Buser and P. Sarnak constructed in [7] hyperbolic surfaces with injectivity radius $\sim \log(g)$ at every point. We improve Theorem 2.1.2 by showing the following result.

Throughout this paper for a positive real number R, we denote by $\lceil R \rceil$ the smallest integer greater or equal to R.

Theorem IV. *Let M be a closed Riemannian surface of genus $g \geq 2$. There are at least $\lceil \log(2g) + 1 \rceil$ homotopically independent loops $\alpha_1, \ldots, \alpha_{\lceil \log(2g)+1 \rceil}$ based at the same point in M, such that for every $i \in \{1, ..., \lceil \log(2g) + 1 \rceil\}$,*

$$\text{length}(\alpha_i) \leq C \frac{\log(g)}{\sqrt{g}} \sqrt{\text{Area}(M)},$$

where C is a universal constant independent from the genus.

Theorem IV substantially improves Theorem 2.1.2. Under the same hypothesis as Theorem 2.1.2, Theorem IV guarantees the existence of $\lceil \log(2g) + 1 \rceil$ homotopically independent loops based at the same point (instead of two) of length roughly bounded by $\log(g)$ (instead of g^α). Note that, if the homotopical systole of the surface M in Theorem IV is bounded away from zero, then the $\lceil \log(2g) + 1 \rceil$ loops can be even chosed to be homologically independent (see Theorem 2.4.2). Also Theorem IV recaptures the following result by S. Sabourau.

Theorem 2.1.3 (Sabourau, [25]). *There exists a positive constant C such that every closed Riemannian surface M of genus $g \geq 2$ and area normalized to g, satisfies*

$$\mathrm{sys}_0(M) \leq C \log(g),$$

where $\mathrm{sys}_0(M)$ is denifed as the length of the shortest non-contractible loop in M which is trivial in $H_1(M, \mathbb{Z})$.

Note that Sabourau splits his proof into two cases. In the first case, he supposes that $\mathrm{sys}_0(M) \leq 4\,\mathrm{sys}(M)$ and then he deduces the result from Gromov's $\log(g)$ bound on the systole. Meanwhile, Theorem IV provides a unified proof of this theorem without refering to Gromov's asymptotic systolic inequality.

Gromov's $\log(g)$ bound on the systole has an analog for metric graphs. Note that for a metric graph Γ, the homotopical systole coincides with the homological systole. We will denote it by $\mathrm{sys}(\Gamma)$. The best bound on the systole of a metric graph is due to B. Bollobàs, E. Szemerédi and B. Thomason [3], [4]. Specifically, they proved that the systole of every connected metric graph of first Betti number $b \geq 2$, and length normalized to b is at most $4\log(b+1)$.

Exactly as for surfaces, given a metric graph of first Betti number $b \geq 2$ and of length normalized to b, one might wonder about the number of homologically independent loops based at the same point satisfying the B. Bollobàs, E. Szemerédi and B. Thomason $\log(b)$ bound. We answer this question here.

Theorem V. *Let Γ be a connected metric graph of first Betti number $b \geq 2$ and of length normalized to b. Let $n \in \{1, \dots, b\}$. There exist at least n homologically independent loops in Γ based at the same point and of length at most $24(\log(b) + n)$.*

An interesting value of n is $n = \lfloor \log(b) \rfloor$, *i.e.*, the integral part of $\log(b)$. In this case, Theorem V asserts that for every connected metric graph Γ of first Betti number $b \geq 2$ and of length b, there exist at least $\lfloor \log(b) \rfloor$ homologically independent loops based at the same point of length at most $48\log(b)$. This extends B. Bollobàs, E. Szemerédi and B. Thomason $\log(b)$ bound on the homological systole of Γ to $\lfloor \log(b) \rfloor$ homologically independent loops of Γ based at the same point.

One might wonder how far from being optimal Theorem V is. We show that it cannot be substantially improved. Indeed, let b and n be two integers such that $b \geq 2$ and $1 \leq n \leq b$. There exists a connected metric graph of first Betti number b and length normalized to b, such that there are at most $\lfloor 24(\log(b)+n) \rfloor + 1$ homologically independent loops in Γ

based at the same point of length at most $24(\log(b)+n)$ (*cf.* Theorem 2.3.2). In particular, this result shows that for $n \geq \lceil \log(b) \rceil$, there exists a connected metric graph Γ of first Betti number and length normalized to b, such that there are at most $52n$ homologically independent loops in Γ based at the same point of length at most $24(\log(b) + n)$.

This paper is organised as follows. In Section 2.2, we show the benefits of short homotopically independent loops based at the same point. In Section 2.3, we give the proof of Theorem V. In Section 2.4, we show how to extend Theorem V to closed surfaces with systole bounded away from zero. In Section 2.5, we show that on a given closed surface the cut locus of a simple closed geodesic captures its topology. In Section 2.6, we prove Theorem IV.

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2.2 Benefits of short homotopically independent loops based at the same point

In this section, we show two applications of homotopically independent loops based at the same point of bounded length.

Let M be a closed Riemannian surface of genus $g \geq 2$. If α and β are two homotopically independent loops based at the same point in M, then

$$\mathrm{sys}_0(M) \leq \mathrm{length}(\alpha\beta\alpha^{-1}\beta^{-1}).$$

In particular, if $\sup(\mathrm{length}(\alpha), \mathrm{length}(\beta)) \leq C \log(g)$, then

$$\mathrm{sys}_0(M) \leq 4C \log(g).$$

Notice that the above observation allows us to recapture the result of Theorem 2.1.3 on the separating systole by means of Theorem IV. Also we would like to point out that Gromov's upper bound $C_\alpha \, g^{1-\alpha}$ on the length of two homotopically independent loops based at the same point in Theorem 2.1.2 is not sufficient to prove that the length of the separating systole of a closed Riemannian surface of genus $g \geq 2$ and area g is bounded above by $\sim \log(g)$.

Another use of homotopically independent loops based at the same point v of a closed Riemannian surface M, is to contribute to the area of balls centered at a lift $\widetilde{v}$ of v in the universal cover $\widetilde{M}$ of M. Let us clarify this idea here. Consider a system $S = \{\alpha_1, \ldots, \alpha_k\}$ of pairwise non-homotopic loops based at v. Let

$$L = \sup_{1 \leq i \leq k} \mathrm{length}(\alpha_i).$$

Denote by s half the systole of M at the point v, *i.e.* half the length of the shortest non contractible loop based at v. Let H'_r (resp N_r) be the set of elements of $H = \langle S \rangle$ (resp $\pi_1(M, v)$) of length less than r, where the length of $\alpha \in \pi_1(M, v)$ is defined as $\mathrm{length}(\alpha) = \mathrm{dist}(\widetilde{v}, \alpha.\widetilde{v})$. It is the minimal length of a loop based at v representing α. Let $R > s + L$. Consider the ball $B = B_{\widetilde{M}}(\widetilde{v}, r_0)$, where $r_0 = R - s$. Every element γ_i of N_{r_0} yields a point $\widetilde{v}_i = \gamma_i.\widetilde{v}$ in B. The balls $B_{\widetilde{M}}(\widetilde{v}_i, s)$ are disjoint and of the same area. We have

$$\mathrm{Area}\, B_{\widetilde{M}}(\widetilde{v}, R) \geq \mathrm{card}(N_{r_0})\, \mathrm{Area}\, B_M(v, s), \tag{2.2.1}$$

where $\mathrm{card}(N_{r_0})$ is the cardinal of N_{r_0}.
Also notice that

$$\mathrm{card}(N_{r_0}) \geq \mathrm{card}(H'_{r_0}). \tag{2.2.2}$$

Thus, a lower bound on the cardinal $\mathrm{card}(H'_{r_0})$ of H'_{r_0} yields also a lower bound on $\mathrm{card}(N_{r_0})$. One way to bound $\mathrm{card}(H'_{r_0})$ from below is the following. We define a norm $\|\ \|$ on H as follows. For β in H, we define the word length $\| \beta \|$ of β as the smallest integer n such that $\beta = \gamma_1 \dots \gamma_n$ where $\gamma_i \in S \cup S^{-1}$. Denote by H^w_r the set of elements of H of word length less than r. We have

$$\mathrm{card}(N'_r) \geq \mathrm{card}(H^w_{r/L}). \tag{2.2.3}$$

Combining (3.1), (3.2) and (3.3) we got

$$\mathrm{Area}\, B_{\widetilde{M}}(\widetilde{v}, R) \geq \mathrm{card}(H^w_{r/L})\, \mathrm{Area}\, B_M(v, s). \tag{2.2.4}$$

Now let $r' > 1$. Notice that $H^w_{r'}$ is maximal if H is free of rank k. That is guaranteed if the loops $\alpha_1, \dots \alpha_k$ are homotopically independent in M. It is now clear how homotopically independent loops based at the same point v contribute to the area to the balls centered at points in the fiber over v in $\widetilde{M}$ whenever the radii R of these balls is longer than $s + L$. Moreover, since R must be at least $s + L$, it is straightforward to see that the shorter the L, the better the result. This means that the upper bound of the lengths of the α_i's is also important.

2.3 Short homologically independent loops on graphs

In this section we prove Theorem V. Recall that this theorem extends the Bollobàs-Szemerédi-Thomason $\log(b)$ bound on the homological systole of graphs to $\lceil \log(g) \rceil$ homologically independent loops based at the same point.

First let us recall some definitions. By definition, a graph Γ is a finite one-dimensional CW-complex (multiple edges and loops are allowed). The first Betti number of a graph Γ can be computed as follows :

$$b(\Gamma) = e - v + n,$$

where e, v and n are respectively the number of edges, vertices and connected components of Γ. A metric graph (Γ, h) is a graph endowed with a length space metric h. The length of a subgraph of Γ is its one-dimensional Hausdorff measure. For more details on graphs we refer the reader to [8].

Definition 2.3.1. *Let Γ be a connected graph of first Betti number $b \geq 1$. A family of loops $(\alpha_1, ...\alpha_k)$ in Γ is said to be* homologically independent *if their homology classes in $H_1(\Gamma, \mathbb{R})$ are.*

Note that this definition extends also to closed Riemannian manifolds.

Now we prove Theorem V.

Theorem 2.3.1. *Let Γ be a connected metric graph of first Betti number $b \geq 2$ and of length normalized to b. Let $n \in \{1, \dots, b\}$. There exist at least n homologically independent loops in Γ based at the same point and of length at most $24(\log(b) + n)$.*

Proof. By definition of the first Betti number b, there exist b homologically independent loops $\alpha_1, \dots, \alpha_b$ in Γ. Fix a point x of α_1. For $i = 1, \dots, b$, let C_i be a minimizing curve from x to α_i. We have $\text{length}(C_i \alpha_i {C_i}^{-1}) \leq \text{length}(C_i) + \text{length}(\alpha_i) + \text{length}(C_i)$. Notice that $\text{length}(C_i) + \text{length}(\alpha_i) \leq b$. Thus, there exists b homologically independent loops in Γ based at the same point of length at most $2b$ ($\leq 24(\log(b) + \frac{b}{2})$). This yields the desired result for $n \in \{\frac{b}{2}, b\}$. Now we consider the case when $n < \frac{b}{2}$. In particular, we suppose $b \geq 3$. By a short cycle of Γ we mean a simple loop of length at most $12 \log(b)$. Let X be a maximal set of homologically independent short cycles of Γ and denote by N its cardinal. We claim that

$$N \geq \frac{b}{2}.$$

Indeed, we construct $k = \lceil \frac{b}{2} \rceil$ graphs $\Gamma_k \subset \dots \subset \Gamma_1 = \Gamma$ and k simple loops as follows. Remove an edge from a systolic loop γ_1 of Γ_1 and denote by Γ_2 the resulting graph. The graph Γ_2 is connected and of first Betti number $b_2 = b - 1$. Now remove an edge from a systolic loop γ_2 of Γ_2 and denote by Γ_3 the resulting graph. By induction, we keep doing this until we get Γ_k. From the inequality (1.1) and since $k = \lceil \frac{b}{2} \rceil$ we have for every $i = 1, \dots, k$,

$$\begin{aligned} \text{length}(\gamma_i) &\leq 4 \frac{\log(1 + b - i + 1)}{b - i + 1} \text{length}(\Gamma_i) \\ &\leq 12 \log(b). \end{aligned}$$

By construction, the k loops $\{\gamma_i\}_{i=1}^k$ are homologically independent in Γ. So the claim is proved.

We divide the set X as follows. Take any element α_1 of X and denote by Y_1 the set $\{\beta \in X \mid \text{dist}(\beta, \alpha_1) \leq 4n\}$. Let α_2 be an element of $X \setminus Y_1$ and denote by Y_2 the set $\{\beta \in X \mid \text{dist}(\beta, \alpha_2) \leq 4n\}$. By induction we continue this process which eventually ends since X is finite. Let $\alpha_j \in X$ be the last short cycle obtained from this process, *i.e.*, let α_j be an element of $X \setminus Y_1 \cup ... \cup Y_{j-1}$ such that $Y_1 \cup \dots \cup Y_{j-1} \cup Y_j = X$. For $i = 1, \dots, j$, we denote by T_i the cardinal of Y_i. We claim that there exists an i_0 such that

$$T_{i_0} \geq n.$$

Indeed, suppose the opposite. We have

$$\frac{b}{2} \leq N = \text{card}(X) \leq \sum_{i=1}^{j} T_i < jn.$$

So $j > \frac{b}{2n} > 1$. For $i \neq i'$, we have $\operatorname{dist}(\alpha_i, \alpha_{i'}) > 4n$. This means that the open neighborhoods of radius $2n$ around the $\alpha_i' s$ are pairwise disjoint. Since Γ is connected, the length of the neighborhood of radius $2n$ around each short cycle α_i is at least $\operatorname{length}(\alpha_i) + 2n$. This implies that

$$\operatorname{length}(\Gamma) > 2nj > b.$$

Hence a contradiction. So there is an i_0 such that $T_{i_0} \geq n$.
Now fix a vertex a of α_{i_0} and let β be any element of $Y_{i_0} \setminus \{\alpha_{i_0}\}$. Let b and c be two vertices of α_{i_0} and β respectively such that $\operatorname{dist}(\alpha_{i_0}, \beta) = \operatorname{dist}(b, c)$. Also, let C_{ab} be a minimizing curve from a to b and C_{bc} be a minimizing curve from b to c. The following holds.

- $\operatorname{length}(C_{ab}) \leq \operatorname{length}(\alpha_{i_0})/2$
- $\operatorname{length}(C_{bc}) \leq 4n$.

The loop $\beta' = C_{ab}C_{bc}\beta C_{cb}C_{ba}$ is homologuous to β and satisfies

$$\operatorname{length}(\beta') \leq 24\log(b) + 8n.$$

So the T_{i_0} short cycles of Y_{i_0} give rise to T_{i_0} homologically independent loops of Γ based at the same point a and of length at most $24(\log(b) + n)$. □

Corollary 2.3.1. *Let Γ be a connected metric graph of first Betti number $b \geq 2$. Let $n \in \{1, \ldots, b\}$. There exist at least n homologically independent loops in Γ based at the same point of length at most $24(\log(b) + n)\, \frac{\operatorname{length}(\Gamma)}{b}$.*

Before stating our next theorem, we construct a connected metric graph $\Gamma_\star$ that will be useful to the rest of this section. Let m and p be two positive integers with $m \geq p$. Denote by q and r the quotient and the remainder in the division of m by p, that is, $m = pq + r$ with $r \in \{0, \ldots, p-1\}$. Also let L and l be two positive constants.
Fix a vertex v. We construct q bouquets $X_1, \ldots, X_q$ of p circles and a bouquet X_{q+1} of r circles. We define $\Gamma_\star$ by joining the vertex of each bouquet X_i to the vertex v by an edge w_i. See Figure 1. We define a metric h on $\Gamma_\star$ such that $(\Gamma_\star, h)$ is a length metric space as follows. For $i = 1, \ldots, q$, set $\operatorname{length}(w_i) = L$, and $\operatorname{length}(X_i) = l$. Also set $\operatorname{length}(X_{q+1}) + \operatorname{length}(w_{q+1}) = r$. It is straightforward to see that the graph $\Gamma_\star$ is connected, of first Betti number m and of length $q(L + l) + r$. We claim that there are at most $p + r$ $(\leq 2p - 1)$ homologically independent loops based at the same point of length at most $2L$. Indeed, notice that there exist at most r homologically independent loops based at v of length less than $2L$. So let m be any point of $\Gamma_\star$ other than the point v. There exists a unique i such that $m \in X_i \cup w_i$. Now notice that if we want to find more than $p + r$ homologically independent loops based at m, one of them must cross at least two times one of the edges w_j, with $j \in \{1, \ldots, q\} \setminus \{i\}$. Thus, the length of this loop exceeds $2L$.

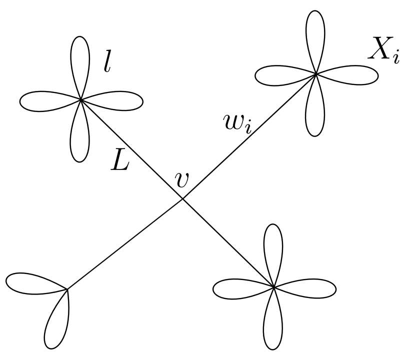

FIGURE 2.1 – The graph $\Gamma_\star$ for $m = 12$, $p = 4$, $q = 3$ and $r = 2$.

Our next theorem shows that one cannot substantially improve Theorem 2.3.1, thus it is roughly optimal.

Theorem 2.3.2. *Let b and n be two integers such that $b \geq 2$ and $1 \leq n \leq b$. Let $\lambda > 0$. There exists a connected metric graph of first Betti number b, of length normalized to b, such that there are at most $\lfloor \lambda(\log(b)+n) \rfloor + 1$ homologically independent loops in Γ based at the same point of length at most $\lambda(\log(b) + n)$*

Proof. We only need to consider the case when $b \geq \lfloor \lambda(\log(b) + n) \rfloor + 1$ since the other case is trivial. Denote by q and r respectively the quotient and the remainder in the division of b by $\lfloor \frac{\lambda}{2}(\log(b) + n) \rfloor + 1$. Let $\varepsilon > 0$ be such that

$$\lfloor \frac{\lambda}{2}(\log(b) + n) \rfloor + 1 = \frac{\lambda}{2}(\log(b) + n) + \varepsilon.$$

Consider the graph $\Gamma_\star$ given by the previous construction with

- $m = b$,
- $p = \lfloor \frac{\lambda}{2}(\log(b) + n) \rfloor + 1$,
- $L = \frac{\lambda}{2}(\log(b) + n)$,
- $l = \varepsilon$.

The graph $\Gamma_\star$ is connected, of first Betti number b, of length b and has at most $\lfloor \lambda(\log(b) + n) \rfloor + 1$ homologically independent loops based at the same point of length at most $\lambda(\log(b) + n)$. □

2.4 Short homologically independent loops on surfaces with homotopical systole bounded from below.

In this section we combine ideas from [1] and [17] to extend Theorem 2.3.1 to closed surfaces with systole bounded below.

Definition 2.4.1. *Let (M,h) be a closed Riemannian surface of genus g. The image in M of an abstract graph by an embedding will be referred to as a graph in M. The metric h on M naturally induces a metric on a graph Γ in M. Despite the risk of confusion, we will also denote by h such a metric on Γ.*

Proposition 2.4.1. *Let (M,h) be a closed Riemannian surface of genus $g \geq 1$. Suppose that the homotopical systole of M is at least ℓ. Then, there exists a graph Γ in M such that*

1. *the inclusion map $i : \Gamma \to M$ is distance non-increasing ;*
2. *the homomorphism $i_* : H_1(\Gamma, \mathbb{R}) \to H_1(M, \mathbb{R})$ induced by the inclusion is an isomorphism ;*
3. $$\text{length}(\Gamma) \leq \frac{2^9 \,\text{Area}(M,h) + g}{\min\{1, \ell\}}.$$

Proof. Without loss of generality, we suppose that $\ell \leq 1$. This proposition is the same as Proposition 6.1 in [17], where ℓ was taken to be $\frac{1}{2}$ and the area is equal to $\frac{1}{2^{11}}(2g-1)$ instead of g. The proof of Proposition 6.1 in [17] starts by fixing $r_0 = \frac{1}{2^5}$, In our case we fix $r_0 = \frac{\ell}{2^4}$ and reproduce the argument. □

Before stating out next theorem, let us recall the following theorem.

Theorem 2.4.1 ([15]). *Let M be a closed Riemann surface of Euler characteristic $\chi(M) \leq 0$. Any subgroup of $\pi_1(M)$ generated by k elements, where $k < 2 - \chi(M)$, is a free group.*

Now we can prove the following result.

Theorem 2.4.2. *Let M be a closed orientable Riemannian surface of genus $g \geq 1$ with homotopical systole at least ℓ and area normalized to g. Let $n \in \{1, \dots, 2g\}$ be an integer. There exist at least n homologically independent loops $\gamma_1, ..., \gamma_n$ based at the same point in M such that for every $i = 1, \dots, n$, we have*

$$\text{length}(\gamma_i) \leq 24 C_\ell(\log(2g) + n),$$

where $C_\ell = \frac{2^9}{\min\{1,\ell\}}$.
Moreover, if $n < 2g$ then $\langle \gamma_1, ..., \gamma_n \rangle$ is free of rank n.

Proof. Let Γ be a graph in M that satisfies (1), (2) and (3) of Proposition 2.4.1. The first Betti number of Γ is $2g$. By Corollary 2.3.1, there are at least n homologically independent loops in Γ based at the same vertex of length at most $24C_\ell(\log(2g) + n)$. The images of these loops by the inclusion map i yield the desired loops. The second assumption follows from Theorem 2.4.1. □

Remark 2.4.1. *A non-orientable version of Theorem 2.4.2 holds. Let M be a closed non-orientable surface of genus $g \geq 1$ with homotopical systole at least ℓ and area normalized to g. Let $n \in \{1, \dots, g\}$. There are at least n loops $\gamma_1, ...\gamma_n$ based at the same point v in M whose homology classes in $H_1(M, \mathbb{Z}_2)$ are independent such that for every $i = 1, ..., n$, we have*

$$\text{length}(\gamma_i) \leq 24C'_\ell(\log(g) + n),$$

where $C'_\ell = \frac{C}{\min\{1,\ell\}}$ for some positive constant C. Moreover, if $n < g$ then $\langle \gamma_1, ..., \gamma_n \rangle$ is free of rank n.

2.5 Cut loci and capturing the topology

In this section we extend the notion of cut locus defined originally for points in a Riemannian manifold to simple closed geodesics (this might be already defined but the author didn't find a reference in the literature) and we give some basic results for the new notion.

Let M be a closed surface and p be a point in M. The cut point of p along a geodesic C_p starting at p is the first point $q \in C_p$ such that the arc of C_p between p and any point r on C_p after q is no longer minimizing. The set $\text{Cut}(p)$ of all cut points along all the geodesics issued from p is called the cut locus of p. We extend this notion to simple closed geodesics as follows.

Let $\alpha : [0, l] \to M$ be a simple closed geodesic in M and β be another geodesic that starts orthogonally from α at some point p. The cut point of α along β is the first point $q \in \beta$ such that, for any point r on β beyond q the length of the arc of β between p and r no longer agrees with the distance from r to α. The set $\text{Cut}(\alpha)$ of all the cut points of all the geodesics issued orthogonally from α is called the cut locus of α. An alternative useful way to view $\text{Cut}(\alpha)$ is the following. Denote by $N\alpha$ the normal bundle to α. Each vector $v_t \in N\alpha$ gives rise to a geodesic C_t starting at $\alpha(t)$ such that ${C_t}'(0) = v_t$. Denote by q_t the cut point of α along the geodesic C_t. The point q_t is the image by the exponential map of some vector v'_t parallel to v_t. Let N_1 be the set of the vectors v'_t and N_2 be the set of the vectors $\lambda v'_t$, where $\lambda \in [0, 1)$. Then, $\text{Cut}(\alpha) = \exp(N_1)$.

Lemma 2.5.1.

$$M = \exp(N_1) \cup \exp(N_2),$$

where the union is disjoint.

Proof. Let x be a point in M. There exists a minimizing geodesic ${\sigma_x}^{-1}$ from x to α parametrized by arc length such that $\text{length}({\sigma_x}^{-1}) = \text{dist}(x, \alpha)$. The geodesic ${\sigma_x}^{-1}$ hits α orthogonaly in a point $\alpha(t)$ (*cf.* [9]). Since σ_x is minimizing, the point x is not after the cut point of α along σ_x. That means that the vector $\text{dist}(\alpha(t), x)\sigma'(0) \in N_1 \cup N_2$. Notice that $x = \exp(\text{dist}(\alpha(t), x)\sigma'(0))$. Thus, $M = \exp(N_1) \cup \exp(N_2)$.
Now let us prove that the union is disjoint. Let $y \in \exp(N_1) \cap \exp(N_2)$. Since $y \in \exp(N_2)$, there exists a minimizing geodesic $\sigma_y : [0, \ell] \to M$ from α to y, parametrized by arc

length such that σ_y is still minimizing for some time after y i.e. there exists an $\varepsilon > 0$ such that $\sigma_y : [0, \ell + \varepsilon]$ is a minimizing geodesic from α to $\sigma_y(\ell + \varepsilon)$. On the other hand, since $y \in \exp(N_1)$, there exists a minimizing geodesic δ_y from α to y parametrized by arc length such that δ_y is no longer minimizing after y. Let ϕ be the curve defined by $\phi(t) = \delta_y(t)$ if $t \in [0, \ell]$, and $\phi(t) = \sigma_y(t)$ for $t \in [\ell, \ell + \varepsilon]$. Let $0 < \varepsilon' < \varepsilon$. There exists a minimizing geodesic from $\phi(\ell - \varepsilon')$ to $\phi(\ell + \varepsilon')$ which is of length strictly less than the arc of ϕ between these two points since ϕ is not smooth at $\phi(\ell)$. We conclude that $\mathrm{dist}(\sigma_y(\ell + \varepsilon'), \alpha)$ is strictly less than the length of σ_y between $\sigma_y(0)$ and $\sigma_y(\ell + \varepsilon')$. Hence a contradiction. So the proof is finished. □

Lemma 2.5.2. *The set* $\mathrm{Cut}(\alpha)$ *is a deformation retract of* $M \setminus \{\alpha\}$. *We will say that* $\mathrm{Cut}(\alpha)$ *captures the topology of* $M \setminus \{\alpha\}$.

Proof. Let x be a point of M not in α or $\mathrm{Cut}(\alpha)$. Denote by σ_x the unique minimizing geodesic from x to α. Let x' be the cut point of α along the geodesic σ_x. Clearly, $x' \in \mathrm{Cut}(\alpha)$. Now we can shrink $M \setminus \{\alpha\}$ to $\mathrm{Cut}(\alpha)$ by sliding each point x of M not in α or $\mathrm{Cut}(\alpha)$ to $\mathrm{Cut}(\alpha)$ along the arc of the geodesic σ_x between x and x'. □

Proposition 2.5.1. *Let* (M, g) *be a closed real analytic Riemannian surface and* α *be a simple closed geodesic in* M. *Then* $\mathrm{Cut}(\alpha)$ *is a finite graph.*

We omit the proof of Proposition 2.5.1 since it is essentially the same proof as in [23] p.97.

2.6 Short Homotopically Independent loops on Riemannian Surfaces

In this section we prove Theorem IV. Before doing that, let us give some definitions and some independent propositions that will be useful to the rest of this section.

Lemma 2.6.1. *Let* $F = \langle a, b \rangle$ *be a free subgroup of rank 2 of the fundamental group of a closed Riemannian manifold. For every integer* $n \geq 1$, *the subgroup* $H = \langle b, a^1 b a^{-1}, \ldots, a^{n-1} b a^{-(n-1)} \rangle$ *of* F *is free of rank* n. *Moreover, if* $\mathrm{length}(a) = l_a$ *and* $\mathrm{length}(b) = l_b$. *Then,*

$$\sup_{0 \leq i \leq n-1} \mathrm{length}(a^i b a^{-i}) \leq 2(n-1) l_a + l_b.$$

Proof. Since the subgroup of a free group is free then H is free. Next, we claim that the generator $a^p b a^{-p}$ is not an element of the free subgroup G generated by the elements $a^q b a^{-q}$ for $q \in \{0, \ldots, n-1\} \setminus \{p\}$. Indeed, a reduced word in G starts with a^q with $q \neq p$. So H is of rank n. The length inequality is immediate. □

Proposition 2.6.1. *Let* (M, g) *be a compact Riemannian cylinder. Denote by* α *and* β *the two boundary components of* M. *Suppose that*

$$\mathrm{length}(\alpha) < 1 < \mathrm{length}(\beta).$$

Then there exists a non-contractible simple loop γ *in* M *of length 1 such that the systole of the cylinder* R_γ *bounded by* β *and* γ *is equal to 1.*

In particular, the loop γ *is a systolic loop of* R_γ.

Proof. Let $X = \{\sigma$ simple non-contractible loop in M such that $\mathrm{sys}(R_\sigma) = 1\}$, where by R_σ we mean the cylinder of boundary components β and σ. Clearly the set X is non empty. Let $\ell = \inf_{\sigma \in X} \mathrm{length}(\sigma)$ and ε be a small positive constant. By the definition of the infimum, there exists a simple non-contractible loop σ_0 such that $\mathrm{sys}(R_{\sigma_0}) = 1$ with $\ell \leq \mathrm{length}(\sigma_0) \leq \ell + \varepsilon$. The systolic loop γ of R_{σ_0} is a simple non-contractible loop in M. Moreover, we have $R_\gamma \subset R_{\sigma_0}$. Thus

$$1 = \mathrm{sys}(R_{\sigma_0}) \leq \mathrm{sys}(R_\gamma) \leq \mathrm{length}(\gamma) = 1.$$

So $\mathrm{sys}(R_\gamma) = 1$. This finishes the proof. □

In the proof of Theorem 2.6.1 below, we will need the following definition.

Definition 2.6.1. *Let M be a closed Riemann surface of genus g (with possibly one disk removed). It is well known that such a surface can be obtained from a polygon P (with possibly one disk removed) by pairwise identifications of its sides where all the vertices of P get identified to a single point on x of M. Such a polygon, will be called a normal representation of M. After identification, the edges of P give rise to $2g$ simple loops (in case M is orientable) or to g simple loops (in case M is non-orientable) based at x and intersecting each other only at x. Such set of loops is called a* canonical system of loops.

Now we prove Theorem IV.

Theorem 2.6.1. *Let M be a closed orientable Riemannian surface of genus $g \geq 2$. There are at least $n = \lceil \log(2g) + 1 \rceil$ homotopically independent loops $\alpha_1, \ldots, \alpha_n$ based at the same point such that for all $i = 1, \ldots, n$,*

$$\mathrm{length}(\alpha_i) \leq 2^{20} \frac{\log(g)}{\sqrt{g}} \sqrt{\mathrm{Area}(M)}.$$

Proof. [Proof of Theorem 2.6.1] Since every smooth metric can be approximated by a real analytic one, we can assume that M is a real analytic Riemannian surface. Multiplying the metric by a constant if needed, we can suppose that the area of M is normalized to g. We only need to consider the case where the homotopical systole of M is less than 1, since the other case is settled down by Theorem 2.4.2. Consider a maximal set X of simple closed geodesics $\alpha_1, ..., \alpha_p$ of length at most 1 which are pairwise disjoint in M and non freely homotopic. Let k be the number of elements of X that are separating. Note that $k \leq p$. The main idea of the proof is to go back to the case where the homotopical systole is at least 1.

Remark 2.6.1. *At first, we were tempted to cut the surface M open along the loops α_i of X and to attach an hemisphere along each of the $2p$ boundary components. This yields at least $k+1$ new closed surfaces $M_1, \ldots, M_{k+1}$, where k is the number of geodesics in X that are separating. We hoped to find the desired loops or two short homotopically independent loops based at the same point in one of the closed surfaces M_i. Recall that the homotopical systole of each M_i is at least 1 so we can use Theorem 2.4.2. Afterwards we wanted to show that these loops do not cross the hemispheres and so lie in the original* *surface M. It doesn't take much time to realize that this idea is naive. One can run into many problems. Let's imagine the case where $p = g$ and all of the geodesics α_i are non-separating like the surface in Figure 2. In this case, the surface obtained by cutting M along the loops α_i and attaching hemispheres is of genus 0 and so the proof collapses. Instead we will cut M along each α_i, chop off some "maximal" cylinders and then glue the boundary components back together to obtain a new surface with systole bounded away from zero.*

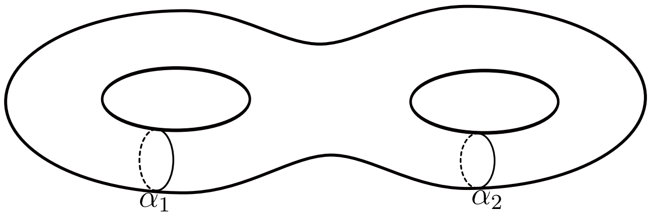

FIGURE 2.2

Let $\varepsilon \in \{-,+\}$. We divide the proof into 5 steps.

Step 1. In this step we chop off cylinders corresponding to short separating loops. If $k = 0$, we skip this step and start directly at the second step. By renumbering the α_i's if needed, we can suppose that for $i = 1, \ldots, k$, the simple closed geodesic α_i is separating. Cut the surface M open along α_1. We obtain two compact surfaces M^- and M^+ with signature $(g-m, 1)$ and $(m, 1)$, where m is some positive integer less than g. Denote by α_1^ε the boundary of the surface M^ε and let S^ε be one of its canonical system of loops. Notice that since the genus of M^ε is at least 1, we have $\text{card}(S^\varepsilon) \geq 2$. We can suppose that for every pair of loops a and b in S^ε, we have $\sup(\text{length}(a), \text{length}(b)) > 1$. Otherwise the proof is finished by Lemma 2.6.1 since a and b do not commute and so generate a free group of rank 2. Cut the surface M^ε open along the loops in S^ε. This gives rise to a cylinder T^ε with two boundary components α_1^ε and β_1^ε such that $\text{length}(\beta_1^\varepsilon) > 1$. So the cylinder T^ε satisfies the hypothesis of Proposition 2.6.1. Thus, there exists a non-contractible simple loop γ_1^ε of length 1 which is a systolic loop of the cylinder R_1^ε bounded by β_1^ε and γ_1^ε is 1. Cut T^ε along γ_1^ε and throw away the cylinder C_1^ε bounded by α_1^ε and γ_1^ε. Now re-glue R_1^ε by pairwise identifications of the edges of β_1^ε. This gives rise to a compact surface M_1^ε with one boundary component γ_1^ε of length 1. Glue the surfaces M_1^- and M_1^+ along their boundaries γ_1^- and γ_1^+. The resulting surface M_1, satisfies the following.
- The surface M_1 has the same genus as the surface M ;
- $\text{Area}(M_1) \leq \text{Area}(M)$;
- A minimal representative in M_1 of the free homotopy class of α_1 is given by the simple loop γ_1 of length 1 obtained by gluing γ_1^- and γ_1^+ together.

Repeat the above process with the $k-1$ remaining elements of X that are separating. This gives rise to a closed surface M_k of the same genus as the surface M such that $\text{Area}(M_k) \leq \text{Area}(M)$. Moreover, any simple closed geodesic of M_k of length less than 1 is non-separating. Perturbing the metric again, we can suppose again that it is a real analytic one.

Step 2. In this step, we chop off cylinders corresponding to short non-separating loops. Cut the surface M_k open along α_{k+1}. This leads to a surface N_k with genus $g-1$ and with two boundary components α^-_{k+1} and α^+_{k+1}. By Lemma 2.5.2, we know that the cut locus $\mathrm{Cut}(\alpha_{k+1})$ of α_{k+1} is a deformation retract of $M \setminus \{\alpha_{k+1}\}$. So the fundamental group of $\mathrm{Cut}(\alpha_{k+1})$ is isomorphic to the fundamental group of N_k. Now cut the surface N_k open along $\mathrm{Cut}(\alpha_{k+1})$. This gives rise to two cylinders. The cylinder T^-_{k+1} with boundary components $(\alpha^-_{k+1}, \beta^-_{k+1})$ and the cylinder T^+_{k+1} with boundary components $(\alpha^+_{k+1}, \beta^+_{k+1})$. Arguing as in Step 1, we can suppose that $\mathrm{length}(\beta^\varepsilon_{k+1}) > 1$. So the cylinder T^ε_{k+1} satisfies the hypothesis of Proposition 2.6.1. Thus there exists a non-contractible simple loop γ^ε_{k+1} of length 1 which is a systolic loop of the cylinder R^ε_{k+1} of boundary components $(\beta^\varepsilon_{k+1}, \gamma^\varepsilon_{k+1})$ is 1. Cut T^ε_{k+1} open along γ^ε_{k+1} and throw away the cylinder C^ε_{k+1} bounded by α^ε_{k+1} and γ^ε_{k+1}. Now re-glue the cylinder R^ε_{k+1} by re-identifying the sides of β^ε_{k+1}. This gives rise to two compact surfaces M^-_{k+1} and M^+_{k+1} with boundary components that can be pairwise identified. Gluing these two surfaces together we get a closed surface M_{k+1} that satisfies the following.

- The surface M_{k+1} has the same genus as the surface M_k.
- $\mathrm{Area}(M_{k+1}) \leq \mathrm{Area}(M_k)$.
- A minimal representative of the free homotopy class of α_{k+1} in M_{k+1} is given by the simple loop γ_{k+1} of length 1, obtained by gluing γ^-_{k+1} and γ^+_{k+1} together.

Repeat the above process with the $p-k-1$ remaining elements of X. This gives rise to a closed surface M_p of the same genus as the surface M such that $\mathrm{Area}(M_p) \leq \mathrm{Area}(M)$.

Before proceeding to the next step, recall that the simple closed geodesics $\alpha_1, \ldots, \alpha_p$ in the original surface M correspond to the simple closed geodesics $\gamma_1, \ldots, \gamma_p$ in the surface M_p. Also recall that the cylinders C^-_i and C^+_i in M share the same boundary component α_i. We denote by C_i the cylinder with boundary components (γ^-_i, γ^+_i), that is, $C_i = C^+_i \cup C^-_i$.

Step 3. In this step, we show that we can suppose that two different cylinders C_j and $C_{j'}$ in M are distant from each other. Specifically, we have $\mathrm{dist}_M(C_j, C_{j'}) > 2^{18}\log(g)$. In other words, we have

$$\mathrm{dist}_{M_p}(\gamma_j, \gamma_{j'}) > 2^{18}\log(g). \tag{2.6.1}$$

Indeed, suppose the opposite. Without loss of generality, suppose that the distance between C_j and $C_{j'}$ is equal to $\mathrm{dist}(\gamma^-_j, \gamma^-_{j'})$. Let z_1 be a point on C_j and z_2 be a point on $C_{j'}$ such that $\mathrm{dist}(z_1, z_2) = \mathrm{dist}(\gamma^-_j, \gamma^-_{j'})$. Consider the loop μ that starts at z_1, travels along a minimizing geodesic between z_1 and z_2, makes a complete tour along $\gamma^-_{j'}$ and then comes back to z_1. We have that $\mathrm{length}(\mu) \leq 2^{19}\log(g) + 1$. Notice also that μ and γ^-_j do not commute. In particular, they are homotopically independent. So by Lemma 2.6.1 (take $a = \gamma^-_j$ and $b = \mu$), the proof of the theorem is finished.

Step 4. In this step, we show that we can suppose that

$$\mathrm{sys}(M_p) \;\geq\; 1.$$

Indeed, by contradiction, suppose that there is a systolic loop μ of M_p of length less than 1. We claim that the geodesic μ transversally intersects at least one of the $\gamma_i's$.

Indeed, suppose the opposite, and denote by μ' the simple closed geodesic in the original surface M that corresponds to μ. Since μ does not transversally intersects any of the $\gamma_i's$, the loop μ' is disjoint from all the cylinders C_i. In particular, μ' does not intersect any of the loops α_i. This contradicts the maximality of X, since length$(\mu') < 1$. Let $j \in \{1, \ldots, n\}$ be such that μ transversally intersects γ_j. That means that in the surface M, the loop μ' goes across the cylinder C_j. Now we claim that μ intersects only one γ_j. Indeed, the length of μ' is less than 1 and the distance between any pair of cylinders C_j and $C_{j'}$ is greater than 1. Therefore, μ intersects only one γ_j. Moreover, the two minimizing simple loops μ and γ_j do not commute.

Lemma 2.6.2. *Let β be a loop in M_p of length less than L that transversally intersects only one geodesic γ_j and does not commute with it. Then there exist two loops a, b based at the same point in the original surface M that do not commute and such that* length$_M(a) = 1$ *and* length$_M(b) \leq 2L+1$. *In particular, the loops a and b are homotopically independent.*

Proof. We give β and γ_j some orientation. Let $x_1, \ldots, x_q$ be the transversal intersection points of β and γ_j counted with multiplicity and ordered in the sense that if we start walking on β, then x_i is the $i-th$ time β intersects γ_j. Suppose that $q \geq 2$ (the case $q = 1$ will be treated in the end of the proof). Let $\beta_{i,i+1}$ be the simple loop based at x_i defined as the concatenation of the oriented arc of β between x_i and x_{i+1} and the oriented arc $c_{i+1,i}$ of γ_j between x_{i+1} and x_i. The loop β is homotopic to the loop $\beta_{1,2}c_{1,2} \ldots \beta_{q,q+1}c_{q,q+1}$, where by convention $c_{i,i+1}$ is the inverse of $c_{i+1,i}$, and $x_{q+1} = x_1$.

Notice from the above equality that at least one of the curves $\beta_{i,i+1}c_{i,i+1}$ does not commute with γ_j, for otherwise we will have that β commute with γ_j, which is a contradiction.

Now let $\beta_{k,k+1}c_{k,k+1}$ be one of the curves $\beta_{i,i+1}c_{i,i+1}$ that does not commute with γ_j. The curve $\beta_{k,k+1}c_{k,k+1}$ is homotopic to $\beta_{k,k+1}$, so in particular $\beta_{k,k+1}$ does not commute with γ_j. Recall that the surface M can be obtained from the surface M_p by cutting along the γ_i's and re-inserting the cylinders C_i. Thus, the loop in M that corresponds to β decomposes into a union of curves whose endpoints lie on one of the two boundary components γ_j^- and γ_j^+ of the cylinder C_j. Denote by x'_k and x'_{k+1} the points in M corresponding to the points x_k and x_{k+1} of $\beta_{k,k+1}$ in M_p. We have two cases.

Case 1. The points x'_k and x'_{k+1} lie both the same boundary component, say γ_j^+.
In this case, let β' be the simple loop in M that corresponds to $\beta_{k,k+1}$ (See Figure 3).

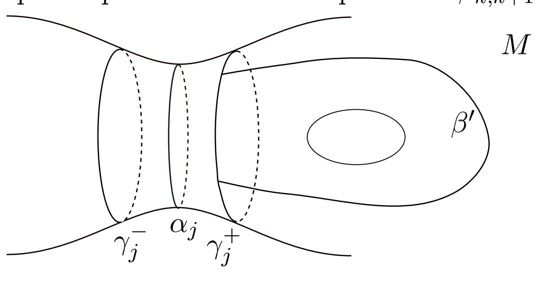

FIGURE 2.3

Take $a = \gamma_j^+$ and $b = \beta'$. These two loops are based at the same point and do not commute. Moreover we have length$(a) = 1$ and length$(b) \leq L + 1$.

Case 2. The points x'_k and x'_{k+1} do not lie both on γ_j^- or γ_j^+.
In this case, let β' be the arc in M that corresponds to the arc of β between x_k and x_{k+1}.

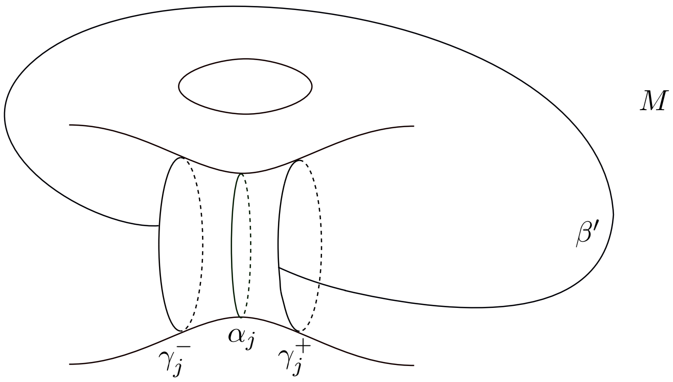

FIGURE 2.4

Take $a = \gamma_j^+$ and $b = \beta'\gamma_j^+\beta'^{-1}$. These two loops are based at the same point and do not commute. Moreover we have length$(a) = 1$ and length$(b) \leq 2L + 1$.

Finally, if the number of intersections $q = 1$, we argue exactly like in case 2 above, supposing that $x_{k+1} = x_k$. That finishes the proof of the Lemma. □

Now, apply Lemma 2.6.2 with $\beta = \mu$ and make use of Lemma 2.6.1 to finish the proof.

Step 5. By Theorem 2.4.2, there are at least $n = \lceil \log(2g) + 1 \rceil$ homotopically independent geodesic loops $\mu_1 \ldots, \mu_n$ based at the same point in M_p with

$$\text{length}(\mu_i) \leq 2^{18} \log(g).$$

If these loops are in the original surface M, *i.e.*, they don't transversally intersect any of the loops γ_i in M_p, then the proof is finished. So suppose the opposite. Let μ be one the loops $\mu_1 \ldots, \mu_n$ that transversally intersects at least one of the γ_i's in M_p. From (6.1), the loop μ (transversally) intersects exactly one loop γ_j in M_p. By Lemma 2.6.2, we show that there exist two loops a, b in the original surface M based at the same point with length$(a) = 1$ and length$(b) \leq 2^{19} \log(g) + 1$. The result follows from Lemma 2.6.1. □

Remark 2.6.2. *Theorem 2.6.1 extends to non-orientable surfaces with multiplicative constant 2^{22} instead of 2^{20} by passing to the double oriented cover.*

Corollary 2.6.1. *There exists a positive constant C such that the separating systole of every closed Riemannian surface M of genus $g \geq 2$ and area g satisfies*

$$\text{sys}_0(M) \leq C \log(g).$$

Proof. From Theorem 2.6.1, there exist two non-commuting loops a and b based at the same point of length at most $c \log(g)$ for some positive constant c. The commutator $[a, b]$ of a and b, of length at most $4c \log(g)$, yields a bound on the separating systole of M. □

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